On a generalization of the Chvátal-Gomory closure

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Abstract. Many practical integer programming problems involve variables with one or two-sided bounds. Dunkel and Schulz (2012) considered a strengthened version of Chvátal-Gomory (CG) inequalities that use 0-1 bounds on variables, and showed that the set of points in a rational polytope that satisfy all these strengthened inequalities is a polytope. Recently, we generalized this result by considering strengthened CG inequalities that use all variable bounds. In this paper, we generalize further by considering not just variable bounds, but general linear constraints on variables. We show that all points in a rational polyhedron that satisfy such strengthened CG inequalities form a rational polyhedron.

Keywords: Integer programming· Cutting planes· Chvátal-Gomory cuts.

1 Introduction

Let $S \subseteq \mathbb{Z}^n$, and let P be a rational polyhedron. Let $\alpha x \leq \beta$ be a valid inequality for P and assume that $\alpha \in \mathbb{Z}^n$. Assume further that S has a point satisfying $\alpha x \leq \beta$. Let

$$\lfloor \beta \rfloor_{S,\alpha} = \max\{\alpha x : x \in S, \alpha x \leq \beta\}.$$

We call the inequality $\alpha x \leq \lfloor \beta \rfloor_{S,\alpha}$ an *S-Chvátal-Gomory cut* for P (or *S-CG cut,* for short). This inequality is valid for $P \cap S$. If $\alpha x \leq \beta$ is valid for P, but S does not contain a point satisfying this inequality, then $P \cap S$ is clearly empty, and we say that $\mathbf{0}x \leq -1$ is an S-CG cut for P derived from $\alpha x \leq \beta$. In a similar manner, we define

$$\lceil \beta \rceil_{S,\alpha} = \min\{\alpha x : x \in S, \alpha x \geq \beta\},$$

assuming that S has a point satisfying $\alpha x \geq \beta$. Then we say that $\alpha x \geq \lceil \beta \rceil_{S,\alpha}$ is the S-CG cut obtained from $\alpha x \geq \beta$. We define the S-CG closure of a polyhedron P to be the set of all points in P that satisfy all S-CG cuts for P, and we denote this set by P_S.

For the case $S = \mathbb{Z}^n$, S-CG cuts for a polyhedron P are essentially the same as Chvátal-Gomory (CG) cuts [4, 11] for P, an important class of cutting planes for integer programming problems. More precisely, let $\alpha x \leq \beta$ be a valid inequality for P such that α is an integral vector. Then $\alpha x \leq \lfloor \beta \rfloor$ is called a CG inequality for P. We also have

$$\lfloor \beta \rfloor \geq \max\{\alpha x : x \in \mathbb{Z}^n, \alpha x \leq \beta\} = \lfloor \beta \rfloor_{\mathbb{Z}^n,\alpha}.$$

The above inequality becomes an equality if the coefficients of α are coprime integers. Therefore, when $S = \mathbb{Z}^n$, P_S is equal to the *Chvátal closure* of P. The Chvátal closure is known to be a rational polyhedron when P is a rational polyhedron (see Schrijver [14]). There exist other classes of convex sets whose Chvátal closure is a rational polyhedron [2, 5, 9]. For general S, the hyperplane $\{x \in \mathbb{R}^n : \alpha x = \beta\}$ is moved until it hits a point in S; the resulting hyperplane is given by $\{x \in \mathbb{R}^n : \alpha x = \lfloor \beta \rfloor_{S,\alpha}\}$. Here, the gap $\beta - \lfloor \beta \rfloor_{S,\alpha}$ can be larger than 1 when $S \neq \mathbb{Z}^n$. In this case, S-CG cuts can be viewed as a special case of *wide split cuts*, where each cut coincides with one side of a *wide split disjunction* (introduced by Bonami, Lodi, Tramontani, and Wiese [1]). Since there is no constant bound on the gap $\beta - \lfloor \beta \rfloor_{S,\alpha}$, standard arguments for proving the polyhedrality of the Chvátal closure cannot be directly applied to the S-CG closure when $S \neq \mathbb{Z}^n$.

Dunkel and Schulz [8] studied S-CG cuts and P_S for the case $P \subseteq [0,1]^n$ and $S = \{0,1\}^n$. Clearly, the inequality $\alpha x \leq \lfloor \beta \rfloor_{S,\alpha}$ dominates the inequality $\alpha x \leq \lfloor \beta \rfloor$ in this case. They proved that when $S = \{0,1\}^n$, the set P_S is a rational polyhedron. The inequalities studied by Dunkel and Schulz are valid for the 0-1 knapsack set $\{x \in \{0,1\}^n : \alpha x \leq \beta\}$; valid inequalities for such knapsack sets are used to solve practical problem instances in Crowder, Johnson, and Padberg [3], and an associated closure operation is defined by Fischetti and Lodi [10]. Pashkovich, Poirrier, and Pulyassary [12] showed that the aggregation closure – which is the set of points satisfying valid inequalities for all knapsack sets $\{x \in \mathbb{Z}^n : x \geq 0, \alpha x \leq \beta\}$ where $\alpha x \leq \beta$ is valid for P and $\alpha \leq 0$ or $\alpha \geq 0$ – is polyhedral for packing and covering polyhedra. For packing polyhedra, Del Pia, Linderoth, and Zhu [13] independently proved the same result.

In [6], we showed that P_S is a rational polyhedron when $P \subseteq \mathrm{conv}(S)$ and S is the set of integral points that satisfy an arbitrary collection of single variable bounds. In this paper, we prove that even if S is the set of integer points contained in an arbitrary rational polyhedron, P_S is a rational polyhedron.

Theorem 1. *Let $S = R \cap \mathbb{Z}^n$ for some rational polyhedron R and $P \subseteq \mathrm{conv}(S)$ be a rational polyhedron. Then P_S is a rational polyhedron.*

The proof outline is as follows. We first prove that the result holds when R is a rational cylinder, via a unimodular mapping of R to the set $T \times \mathbb{R}^l$ where $l \leq n$ and $T \subseteq \mathbb{R}^{n-l}$ is a polytope. The case $R = T \times \mathbb{R}^l$ is already covered in [6]. We then consider the case when R is pointed, and we show, via a translation, that R can be assumed to be contained in its recession cone. The hardest case in [6] is the case when $S = \mathbb{Z}^n_+$ and P is a packing or covering polyhedron contained in $\mathbb{R}^n_+$. Similarly, the case when R is a pointed polyhedron and P behaves like a packing or covering polyhedron with respect to R is the hardest case in this paper. The main technical difference between a pointed polyhedron R and $\text{conv}(\mathbb{Z}^n_+) = \mathbb{R}^n_+$ is that R can have more than n extreme rays and, in particular, the extreme rays can be linearly dependent. Nevertheless, this case can be dealt with by an argument generalizing the argument in [6] for the case $S = \mathbb{Z}^n_+$. Essentially, we prove that given a valid inequality for P (and the associated hyperplane) that yields a nonredundant S-CG cut, the points at which the hyperplane intersects the rays of the recession cone of R are bounded.

1.1 Preliminaries

Given a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$, we define Π_P as the set of all coefficient vectors that define valid, supporting inequalities for P with integral left-hand-side coefficients:

$$\Pi_P = \left\{(\lambda A, \lambda b) \in \mathbb{Z}^n \times \mathbb{R} :\ \lambda \in \mathbb{R}^m_+,\ \lambda b = \max\{\lambda A x : x \in P\}\right\}.$$

Finally, for $\Omega \subseteq \Pi_P$, we define $P_{S,\Omega}$ as follows:

$$P_{S,\Omega} = \bigcap_{(\alpha,\beta)\in\Omega} \{x \in \mathbb{R}^n :\ \alpha x \leq \lfloor \beta \rfloor_{S,\alpha}\}.$$

P_S, the S-CG closure of P, can be formally defined as P_{S,Π_P}. Notice that if $\Gamma \subseteq \Omega \subseteq \Pi_P$, then $P_S \subseteq P_{S,\Omega} \subseteq P_{S,\Gamma}$. Also, if $S \subseteq T$ for some $T \subseteq \mathbb{Z}^n$, then $P_S \subseteq P_T$. Likewise, for any $\Omega \subseteq \Pi_P$, we have $P_{S,\Omega} \subseteq P_{T,\Omega}$ if $S \subseteq T$.

2 Integer points in a general cylinder

In [6], Dash, Günlük, and Lee proved the following:

Theorem 2 ([6, Theorem 3.4]). *Let $S = F \times \mathbb{Z}^l$ for some finite $F \subseteq \mathbb{Z}^{n-l}$ and $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Then P_S is a rational polyhedron.*

We say that a rational polyhedron R is a *rational cylinder* if the recession cone and lineality space of R are the same. For instance, the convex hull of $F\times\mathbb{Z}^l$ for some finite $F \subseteq \mathbb{Z}^{n-l}$ is a rational cylinder. In this section, we consider the case where $S = R \cap \mathbb{Z}^n$ for some rational cylinder R.

Remember that a unimodular transformation is a mapping $\tau : \mathbb{R}^n \to \mathbb{R}^n$ that maps $x \in \mathbb{R}^n$ to $Ux + v \in \mathbb{R}^n$ for some unimodular matrix $U \in \mathbb{Z}^{n\times n}$ and some integral vector $v \in \mathbb{Z}^n$. Note that the inverse mapping $\tau^{-1}(x) = U^{-1}x - U^{-1}v$ is also a unimodular transformation. For $X \subseteq \mathbb{R}^n$, we denote by $\tau(X)$ the image of X under τ. For $\Pi \subseteq \Pi_P$, although Π is not in the space of $\mathbb{R}^n$, we abuse our notation and define $\tau(\Pi)$ as $\{(\alpha U^{-1}, \beta + \alpha U^{-1} v) : (\alpha, \beta) \in \Pi\} \subseteq \Pi_{\tau(P)}$.

Lemma 1 (Unimodular mapping lemma [6]). *Let $S \subseteq \mathbb{Z}^n$ and $P \subseteq \text{conv}(S)$ be a rational polyhedron. Then $\tau(P) \subseteq \text{conv}(\tau(S))$, and for any $\Pi \subseteq \Pi_P$, $\tau(P_{S,\Pi}) = \tau(P)_{\tau(S),\tau(\Pi)}$. In particular, $\tau(P_S) = \tau(P)_{\tau(S)}$.*

Essentially, we will argue that there is a unimodular transformation mapping S to $F \times \mathbb{Z}^l$ for some finite $F \subseteq \mathbb{Z}^{n-l}$.

Theorem 3. *Let $S = R \cap \mathbb{Z}^n$ for some rational cylinder R and $P \subseteq \operatorname{conv}(S)$ be a rational polyhedron. Then P_S is a rational polyhedron.*

Proof. Since $S = R \cap \mathbb{Z}^n$ and R is a rational cylinder, we have $\operatorname{conv}(S) \cap \mathbb{Z}^n = S$ and, for some integer $g \geq 0$, there exist integer vectors $v^1, \ldots, v^g$ such that $\operatorname{conv}(S) = \operatorname{conv}\left\{v^1, \ldots, v^g\right\} + \mathcal{L}$ where $\mathcal{L}$ is the lineality space of $\operatorname{conv}(S)$. Let $P \subseteq \operatorname{conv}(S)$ be a rational polyhedron. There exists a unimodular transformation τ such that $\tau(\mathcal{L}) = \{\mathbf{0}\} \times \mathbb{R}^l$ and $\tau(\operatorname{conv}(S)) = \tau\left(\operatorname{conv}\left\{v^1, \ldots, v^g\right\}\right) + \tau(\mathcal{L})$ where $0 \leq l \leq n$ is the dimension of $\mathcal{L}$. This implies that $\tau(S) = F \times \mathbb{Z}^l$ for some finite $F \subseteq \mathbb{Z}^{n-l}$. Then Theorem 2 implies that $\tau(P)_{\tau(S)}$ is a rational polyhedron, and by Lemma 1, P_S is a rational polyhedron, as required. □

3 Integer points in a pointed polyhedron

In this section, we consider the case when

$$S = R \cap \mathbb{Z}^n \quad \text{where } R \text{ is a rational pointed polyhedron.}$$

Then $\operatorname{conv}(S) \cap \mathbb{Z}^n = S$ and, for some integers $g, h \geq 0$, there exist integer vectors $v^1, \ldots, v^g, r^1, \ldots, r^h \in \mathbb{Z}^n$ such that $\operatorname{conv}(S)$ can be rewritten as

$$\operatorname{conv}(S) = \operatorname{conv}\left\{v^1, \ldots, v^g\right\} + \operatorname{cone}\left\{r^1, \ldots, r^h\right\}. \tag{1}$$

Since R is pointed, $\operatorname{cone}\left\{r^1, \ldots, r^h\right\}$ has to be pointed as well. Let $N_v = \{1, \ldots, g\}$ and $N_r = \{1, \ldots, h\}$. We will show that the S-CG closure of a rational polyhedron $P \subseteq \operatorname{conv}(S)$ is again a rational polyhedron. To simplify the proof, we will reduce this setting to a more restricted setting with additional assumptions on S and P, and we will see that these assumptions make the structure of S and that of P easier to deal with. The first part of Section 3 explains the reduction, and Sections 3.1 and 3.2 consider the narrower case of S and P obtained after the reduction.

The first assumption we make is the following:

$$\operatorname{rec}(\operatorname{conv}(S)) \subseteq \{\mathbf{0}\} \times \mathbb{R}^{n_2}, \quad \operatorname{conv}(S) \subseteq \operatorname{cone}\left\{e^1, \ldots, e^{n_1}, r^1, \ldots, r^h\right\} \tag{2}$$

where n_2 is the dimension of $\operatorname{rec}(\operatorname{conv}(S))$, $n_1 = n - n_2$, and $e^1, \ldots, e^{n_1}$ are unit vectors in $\mathbb{R}^{n_1} \times \{\mathbf{0}\}$. The following lemma justifies the assumption (2).

Lemma 2. *Let $S = R \cap \mathbb{Z}^n$ for some rational pointed polyhedron R. Then there is a unimodular transformation τ so that $T := \tau(S)$ has the property that $\operatorname{conv}(T) \cap \mathbb{Z}^n = T$ and $\operatorname{conv}(T)$ is of the form (1) satisfying (2).*

Proof. As R is pointed, $\operatorname{conv}(S)$ is also pointed and $\operatorname{conv}(S) \cap \mathbb{Z}^n = S$. Let n_2 be the dimension of $\operatorname{rec}(\operatorname{conv}(S))$. Then there is a unimodular transformation u such that $u(\operatorname{rec}(\operatorname{conv}(S))) = \operatorname{rec}(\operatorname{conv}(u(S))) \subseteq \{\mathbf{0}\} \times \mathbb{R}^{n_2}$. Let $\operatorname{rec}(\operatorname{conv}(u(S))) =$

cone $\{r^1, \ldots, r^h\}$. Then it follows that $e^1, \ldots, e^{n_1}, r^1, \ldots, r^h$ span $\mathbb{R}^n$. Therefore, there exists a sufficiently large integer M such that $v + M(\sum_{i=1}^{n_1} e^i + \sum_{j=1}^{h} r^j) \in$ cone $\{e^1, \ldots, e^{n_1}, r^1, \ldots, r^h\}$ for all vertices v of $\mathrm{conv}(u(S))$. Let ν be the undimodular transformation defined by $\nu(x) := x + M(\sum_{i=1}^{n_1} e^i + \sum_{j=1}^{h} r^j)$ for $x \in \mathbb{R}^n$. Then $\mathrm{conv}(\nu(u(S))) \subseteq$ cone $\{e^1, \ldots, e^{n_1}, r^1, \ldots, r^h\}$, and since ν is just a translation, the recession cone of $\mathrm{conv}(\nu(u(S)))$ remains the same as that of $\mathrm{conv}(u(S))$. So, $\tau = \nu \circ u$ is the desired unimodular transformation. $\square$

By Lemma 1, P_S is a rational polyhedron if and only if $\tau(P)_{\tau(S)}$ is a rational polyhedron, so we may assume that S satisfies (2).

The second assumption is on the structure of the polyhedron P. Let P^1 and P^2 be defined as follows:

$$P^1 := P + \mathrm{cone}\left\{e^1, \ldots, e^{n_1}, r^1, \ldots, r^h\right\}, P^2 := P - \mathrm{cone}\left\{e^1, \ldots, e^{n_1}, r^1, \ldots, r^h\right\}.$$

Since $P \subseteq \mathrm{conv}(S) \subseteq$ cone $\{e^1, \ldots, e^{n_1}, r^1, \ldots, r^h\}$, P^1 is pointed and the extreme points of P^1 are contained in $\mathrm{conv}(S)$. Moreover, P^1 can be written as $P^1 = \{x \in \mathbb{R}^n : Ax \geq b\}$ where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ are matrices satisfying

$$Ax \geq \mathbf{0} \text{ for all } x \in \left\{e^1, \ldots, e^{n_1}, r^1, \ldots, r^h\right\} \quad \text{and} \quad b \geq \mathbf{0}. \tag{3}$$

Similarly, P^2 can be written as $P^2 = \{x \in \mathbb{R}^n : Ax \leq b\}$ for some $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ satisfying (3). Essentially, we can focus on the polyhedra of the form $P^{\uparrow}$ or $P^{\downarrow}$:

$$P^{\uparrow} = \{x \in \mathbb{R}^n : Ax \geq b\} \quad \text{or} \quad P^{\downarrow} = \{x \in \mathbb{R}^n : Ax \leq b\} \tag{4}$$

for some $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ satisfying (3). In Sections 3.1 and 3.2, we prove that the following holds:

($\star$) Let $Q \subseteq \mathbb{R}^n$ be a rational polyhedron of the form $P^{\uparrow}$ or $P^{\downarrow}$ as in (4) for some $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ satisfying (3). We further assume that if $Q = P^{\uparrow}$, then $P^{\uparrow} \subseteq$ cone $\{e^1, \ldots, e^{n_1}, r^1, \ldots, r^h\}$ and the extreme points of $P^{\uparrow}$ are contained in $\mathrm{conv}(S)$. Then Q_S is a rational polyhedron.

The following Lemma implies that proving ($\star$) is sufficient to prove the result for $\mathrm{conv}(S)$ being pointed polyhedra.

Lemma 3. *Let $S \subseteq \mathbb{Z}^n$ be such that $\mathrm{conv}(S) \cap \mathbb{Z}^n = S$, $\mathrm{conv}(S)$ is of the form* (1) *satisfying* (2). *Let $P \subseteq \mathrm{conv}(S)$ be a rational polyhedron. Assume that ($\star$) holds. Then P_S is a rational polyhedron.*

The basic idea for proving Lemma 3 is to come up with a series of unimodular transformations so that the setting in Lemma 3 is reduced to the narrow case in ($\star$). We defer the formal proof of Lemma 3 to a full version of this paper.

3.1 Covering polyhedra

In Sections 3.1 and 3.2, we assume that $\mathrm{conv}(S) \cap \mathbb{Z}^n = S$ and $\mathrm{conv}(S)$ is of the form (1) satisfying (2). In this section, we consider polyhedra of the form

$P^{\uparrow}$ as in (4) where $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ satisfy (3). We will prove that if $P^{\uparrow} \subseteq \text{cone}\left\{e^1,\ldots,e^{n_1},r^1,\ldots,r^h\right\}$ and the extreme points of $P^{\uparrow}$ are contained in conv(S), then $P^{\uparrow}{}_S$ is a rational polyhedron. Notice that every valid inequality for $P^{\uparrow}$ is of the form

$$\alpha x \geq \beta \quad \text{where } \alpha x \geq 0 \text{ for all } x \in \left\{e^1,\ldots,e^{n_1},r^1,\ldots,r^h\right\} \text{ and } \beta \geq 0. \tag{5}$$

As we will be dealing with inequalities of the greater or equal to form in this section, we will abuse notation and define $\Pi_{P^{\uparrow}}$ as follows:

$$\Pi_{P^{\uparrow}} = \left\{(\lambda A, \lambda b) \in \mathbb{Z}^n \times \mathbb{R} : \ \lambda \in \mathbb{R}^m_+, \ \lambda b = \min\{\lambda A x : x \in P^{\uparrow}\}\right\}.$$

Given $(\alpha,\beta) \in \Pi_{P^{\uparrow}}$, the S-CG cut obtained from $\alpha x \geq \beta$ is $\alpha x \geq \lceil \beta \rceil_{S,\alpha}$.

To prove that $P^{\uparrow}{}_S$ is a rational polyhedron, we define the notion of "ray-support". Given a vector $\alpha \in \mathbb{R}^n$, we define the *ray-support* of α, denoted r-supp(α), as follows:

$$r\text{-supp}(\alpha) := \left\{j \in N_r : \ \alpha r^j > 0\right\}.$$

If $(\alpha,\beta) \in \Pi_{P^{\uparrow}}$ and $j \in r$-supp(α), then $\alpha r^j \geq 1$. For $j \in r$-supp(α), the ray generated by r^j always intersects the hyperplane $\{x \in \mathbb{R}^n : \alpha x = \beta\}$, and $(\beta/\alpha r^j) r^j$ is the intersection point. Henceforth, $\beta/\alpha r^j$ is referred to as an "intercept" for convenience for $j \in r$-supp(α). Lemma 4 implies that if every nondominated S-CG cut for $P^{\uparrow}$ has bounded intercepts, then $P^{\uparrow}{}_S$ is a rational polyhedron. The following proposition will be useful:

Proposition 1 ([6, Proposition 2.9]). *Let S be a finite subset of $\mathbb{Z}^n$ and $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Let $H \subseteq \mathbb{R}^n \times \mathbb{R}$ be a rational polyhedron that is contained in its recession cone* rec(H) *and let $\Omega = \Pi_P \cap H$. Then, $P_{S,\Omega}$ is a rational polyhedron.*

Lemma 4. *Let M^* be a positive integer, and let*

$$\Pi = \left\{(\alpha,\beta) \in \Pi_{P^{\uparrow}} : \ \beta/\alpha r^j \leq M^* \text{ for all } j \in r\text{-supp}(\alpha)\right\}. \tag{6}$$

Then $P^{\uparrow}{}_{S,\Pi}$ is a rational polyhedron.

Proof. Recall that conv$(S) = \text{conv}\left\{v^1,\ldots,v^g\right\} + \text{cone}\left\{r^1,\ldots,r^h\right\}$. Let S^* be a finite subset of S defined as

$$S^* := S \cap \left(\text{conv}\left\{v^1,\ldots,v^g\right\} + \left\{\mu_1 r^1 + \cdots + \mu_h r^h : 0 \leq \mu_j \leq M^* \text{ for } j \in N_r\right\}\right).$$

As $S^* \subseteq S$, we have $P^{\uparrow}{}_{S^*,\Pi} \subseteq P^{\uparrow}{}_{S,\Pi}$. In fact, we can show that $P^{\uparrow}{}_{S^*,\Pi} = P^{\uparrow}{}_{S,\Pi}$. It suffices to argue that $\lceil\beta\rceil_{S^*,\alpha} = \lceil\beta\rceil_{S,\alpha}$ for every $(\alpha,\beta) \in \Pi$. The details of this are deferred to a full version of this paper. So, it remains to show that $P^{\uparrow}{}_{S^*,\Pi}$ is a rational polyhedron. Note that we can write $\Pi = \bigcup_{I\subseteq N_r} \Pi(I)$ where $\Pi(I) = \{(\alpha,\beta) \in \Pi : \ r\text{-supp}(\alpha) = I\}$. Therefore, $\Pi(I) = \Pi_{P^{\uparrow}} \cap H(I)$ where

$$H(I) = \left\{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R} : \begin{array}{l} \alpha r^j \geq 1 \text{ for } j \in I, \ \alpha r^j = 0 \text{ for } j \in N_r \setminus I, \\ M^* \alpha r^j \geq \beta \text{ for } j \in I \end{array}\right\}.$$

Notice that $H(I) \subseteq \text{rec}(H(I))$. So, by Proposition 1, $P^{\uparrow}{}_{S^*,\Pi(I)}$ is a rational polyhedron. As $P^{\uparrow}{}_{S^*,\Pi} = \bigcap_{I \subseteq N_r} P^{\uparrow}{}_{S^*,\Pi(I)}$, the proof is complete. $\square$

By Lemma 4, it is enough to argue that all nondominated S-CG cut for $P^{\uparrow}$ have "bounded" intercepts, in the sense that these inequalities belong to Π defined in (6). In the end, we will prove that $P^{\uparrow}{}_S = P^{\uparrow}{}_{S,\Pi}$. Recall that $P^{\uparrow}$ is described by the system $Ax \geq b$ consisting of m inequalities, denoted $a_1 x \geq b_1, \ldots, a_m x \geq b_m$. Since $P^{\uparrow}$ is pointed, we know that $m \geq 1$. By (3), for every $i = 1, \ldots, m$, we have $a_i r^j \geq 0$ for $j \in N_r$. Then for any $\lambda \in \mathbb{R}^m_+$, it follows that $r\text{-supp}(\lambda_i a_i) \subseteq r\text{-supp}(\lambda A)$.

Definition 1. *Let $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. The tilting ratio of λ with respect to A is defined as*

$$r(\lambda, A) = \lambda_1 / \lambda_{t(\lambda,A)}$$

where $t(\lambda, A) = \min \left\{ j \in \{1, \ldots, m\} : \bigcup_{i=1}^{j} r\text{-}supp(a_i) = r\text{-}supp\,(\lambda A) \right\}$. In particular, $\lambda_1 \ldots, \lambda_{t(\lambda,A)} > 0$ and $r(\lambda, A) > 0$.

Definition 2. *Let $B = \max\limits_{1 \leq i \leq m} \{b_i\}$ and $D = \sum_{i=1}^{m} a_i \left(\sum_{i=1}^{n_1} e^i + \sum_{j=1}^{h} r^j \right)$. We define $M_1 = 2\,(mB + 2D)$ and $M = \prod_{i=1}^{m-1} M_i$ where*

$$M_i = (2mB \prod_{j=1}^{i-1} M_j)^{i-1} M_1 \quad \textit{for } i = 2, \ldots, m-1.$$

In particular, $M = 1$ if $m = 1$ and $M \geq M_1 \geq 4$ if $m \geq 2$.

Moreover, $(M_i/M_1)^{1/(i-1)} \geq 4$, and thus, $(M_1/M_i)^{1/(i-1)} \leq 1/4$ for all $i \geq 2$.

We will show in Lemma 5 that if $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ has tilting ratio $r(\lambda, A) > M$, then there exists a $\mu \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ that defines an S-CG cut dominating the one defined by λ, but with $\|\mu\|_1 \leq \|\lambda\|_1 - 1$. We will need a result of Dirichlet:

Theorem 4 (Simultaneous Diophantine Approximation Theorem [7]). *Let k be a positive integer. Given any real numbers $r_1, \ldots, r_k$ and $0 < \varepsilon < 1$, there exist integers $p_1, \ldots, p_k$ and q such that $\left| r_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q}$ for $i = 1, \ldots, k$ and $1 \leq q \leq \varepsilon^{-k}$.*

The following technical lemma generalizes Lemma 4.10 in [6]:

Lemma 5. *Let $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ be such that $(\lambda A, \lambda b) \in \Pi_{P^{\uparrow}}$. If $r(\lambda, A) > M$, then there exists $\mu \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ that satisfies the following: (i) $\|\mu\|_1 \leq \|\lambda\|_1 - 1$, (ii) $(\mu A, \mu b) \in \Pi_{P^{\uparrow}}$, and (iii) $\mu A x \geq \lceil \mu b \rceil_{S,\mu A}$ dominates $\lambda A x \geq \lceil \lambda b \rceil_{S,\lambda A}$.*

Proof. After relabeling the rows of $Ax \geq b$, we may assume that $\lambda_1 \geq \cdots \geq \lambda_m$. Let t stand for $t(\lambda, A)$. If $t = 1$, we have $r(\lambda, A) = 1 \leq M$, a contradiction to our assumption. This implies that $t \geq 2$, and thus, $m \geq 2$. Let Δ be defined as

$$\Delta = \min \left\{ \lambda A r^j : j \in r\text{-supp}(\lambda A) \right\}, \tag{7}$$

and let

$$k = \operatorname{argmin}\left\{\lambda A r^j \,:\, j \in r\text{-supp}\,(\lambda A) \setminus \bigcup_{i=1}^{t-1} r\text{-supp}(a_i)\right\}. \tag{8}$$

By the definition of t, it follows that $r\text{-supp}\,(\lambda A) \setminus \bigcup_{i=1}^{t-1} r\text{-supp}(a_i)$ is not empty, and therefore, k is a well-defined index. Moreover, we obtain

$$\Delta \leq \lambda A r^k = \sum_{i=t}^{m} \lambda_i a_i r^k \leq \lambda_t \sum_{i=t}^{m} a_i r^k \leq D\lambda_t \tag{9}$$

where the first inequality is due to (7), the equality holds due to (8), the second inequality follows from the assumption that $\lambda_1 \geq \cdots \geq \lambda_m$, and the last inequality follows from the definition of D given in Definition 2. Notice that as $r(\lambda, A) = \frac{\lambda_1}{\lambda_t} = \frac{\lambda_1}{\lambda_2} \times \cdots \times \frac{\lambda_{t-1}}{\lambda_t} > M \geq M_1 \times \cdots \times M_{t-1}$, there exists $\ell \in \{1, \ldots, t-1\}$ such that

$$\lambda_i/\lambda_{i+1} \leq M_i \text{ for all } i \in \{1, \ldots, \ell - 1\} \quad \text{and} \quad \lambda_\ell/\lambda_{\ell+1} > M_\ell. \tag{10}$$

We now construct the vector $\mu \in \mathbb{R}^m \setminus \{\mathbf{0}\}$. We consider the case $\ell \geq 2$ first. It follows from the Simultaneous Diophantine Approximation Theorem (with $k = \ell - 1$ and $r_i = \lambda_i/\lambda_\ell$ for $i \in \{1, \ldots, \ell - 1\}$) that there exist positive integers $p_1, \ldots, p_\ell$ satisfying

$$\left|\frac{\lambda_i}{\lambda_\ell} - \frac{p_i}{p_\ell}\right| < \frac{\varepsilon}{p_\ell},\ i \in \{1, \ldots, \ell\} \quad \text{and} \quad p_\ell \leq \varepsilon^{-(\ell-1)} \tag{11}$$

where $\varepsilon = (M_1/M_\ell)^{1/(\ell-1)}$. Moreover, for all $i \in \{1, \ldots, \ell - 1\}$, we can assume that $p_i \geq p_{i+1} \geq p_\ell$, as $\lambda_i \geq \lambda_{i+1}$. If $p_i < p_{i+1}$ for some $i \in \{1, \ldots, \ell - 1\}$, then increasing p_i to p_{i+1} can only reduce $|\lambda_i/\lambda_\ell - p_i/p_\ell|$. Now we define $\mu_1, \ldots, \mu_m$ as follows:

$$\mu_i = \begin{cases} \lambda_i - p_i \Delta & \text{for } i \in \{1, \ldots, \ell\}, \\ \lambda_i & \text{otherwise.} \end{cases} \tag{12}$$

If, on the other hand, $\ell = 1$, we define μ as in (12) with $p_1 = 1$.

We can show that $\mu \geq \mathbf{0}$ (see Claim 1 in Appendix A), implying in turn that $\|\mu\|_1 \leq \|\lambda\|_1 - 1$. Then we can prove that $\mu A x \geq \lceil \mu b \rceil_{S,\mu A}$ dominates $\lambda A x \geq \lceil \lambda b \rceil_{S,\lambda A}$ (See Claim 4 in Appendix A). $\square$

Now we are ready to prove that ${P^\uparrow}_S$ is a rational polyhedron. The following theorem extends Theorem 4.11 in [6]:

Theorem 5. *Let*

$$\Pi = \left\{(\alpha, \beta) \in \Pi_{P^\uparrow} \,:\, \beta/\alpha r^j \leq M^* \ \ \textit{for all } j \in r\textit{-supp}(\alpha)\right\}.$$

where $M^* = mBM$. If $P^\uparrow \subseteq \operatorname{cone}\left\{e^1, \ldots, e^{n_1}, r^1, \ldots, r^h\right\}$ and the extreme points of $P^\uparrow$ are contained in $\operatorname{conv}(S)$, then ${P^\uparrow}_S = {P^\uparrow}_{S,\Pi}$, and in particular, ${P^\uparrow}_S$ is a rational polyhedron.

Proof. As $\Pi \subseteq \Pi_{P^\uparrow}$, we have ${P^\uparrow}_S \subseteq {P^\uparrow}_{S,\Pi}$. We will show that ${P^\uparrow}_S = {P^\uparrow}_{S,\Pi}$ by arguing that for each $(\alpha, \beta) \in \Pi_{P^\uparrow}$, there is an $(\alpha', \beta') \in \Pi$ such that the S-CG cut derived from (α', β') dominates the S-CG cut derived from (α, β) on $P^\uparrow$.

Let $\lambda \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ be such that $(\lambda A, \lambda b) \in \Pi_{P^\uparrow}$, and set $(\alpha, \beta) = (\alpha A, \beta b)$. If $\beta/\alpha r^j \leq M^*$ for all $j \in r\text{-supp}(\alpha)$, then $(\alpha, \beta) \in \Pi$ as desired. Otherwise, consider an arbitrary $j \in r\text{-supp}(\alpha)$ such that $\beta/\alpha r^j > M^*$. Let t stand for $t(\lambda, A)$ and note that

$$M^* < \frac{\beta}{\alpha r^j} = \frac{\sum_{i=1}^m \lambda_i b_i}{\sum_{i=1}^m \lambda_i a_i r^j} \leq \frac{\lambda_1 \sum_{i=1}^m b_i}{\lambda_t \sum_{i=1}^t a_i r^j} = r(\lambda, A) \frac{\sum_{i=1}^m b_i}{\sum_{i=1}^t a_i r^j} \leq mB\, r(\lambda, A),$$

where the last inequality follows from the fact that $b_i \leq B$ for all $i \in \{1, \ldots, m\}$ and the fact that $\sum_{i=1}^t a_i r^j \geq 1$ as $\bigcup_{i=1}^t r\text{-supp}\,(a_i) = r\text{-supp}\,(\lambda A)$.

As $M^* = mBM$, we have $r(\lambda, A) > M$. Then, by Lemma 5, there exists a $\mu \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that $\|\mu\|_1 \leq \|\lambda\|_1 - 1$ and the S-CG cut generated by μ dominates the S-CG cut generated by λ for $P^\uparrow$. If necessary, we can repeat this argument and construct a sequence of vectors $\mu^1, \mu^2, \ldots$, with decreasing norms such that each vector in the sequence defines an S-CG cut that dominates the previous one. Therefore, after at most $\|\lambda\|_1$ iterations, we must obtain a vector $\hat{\mu} \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that $r(\hat{\mu}, A) \leq M$ and $(\hat{\mu}A, \hat{\mu}b) \in \Pi$. As $(\hat{\mu}A, \hat{\mu}b) \in \Pi$ and the S-CG cut generated by $\hat{\mu}$ dominates the S-CG cut generated by λ for $P^\uparrow$, we conclude that ${P^\uparrow}_S = {P^\uparrow}_{S,\Pi}$. Moreover, as ${P^\uparrow}_{S,\Pi}$ is a rational polyhedron by Lemma 4, it follows that ${P^\uparrow}_S$ is a rational polyhedron, as desired. □

3.2 Packing polyhedra and general pointed polyhedra

Similarly, we can show that ${P^\downarrow}_S$ is a rational polyhedron, where $P^\downarrow$ is defined as in (4) for some $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ satisfying (3). Unlike $P^\uparrow$, $P^\downarrow$ is not necessarily pointed. Moreover, we do not assume that the extreme points of $P^\downarrow$ are contained in $\text{conv}(S)$. We may assume that $m \geq 1$. Otherwise, $P^\downarrow = \mathbb{R}^n$, and therefore, ${P^\downarrow}_S = \mathbb{R}^n$ is trivially a rational polyhedron. We defer the proof of the following theorem to a full version of this paper.

Theorem 6. *${P^\downarrow}_S$ is a rational polyhedron.*

By Theorems 5 and 6, it follows that ($\star$), stated before Lemma 3, holds. As a consequence of Lemmas 2 and 3, we obtain the main theorem of Section 3.

Theorem 7. *Let $S = R \cap \mathbb{Z}^n$ for some rational pointed polyhedron R and $P \subseteq \text{conv}(S)$ be a rational polyhedron. Then P_S is a rational polyhedron.*

4 The general case

In this section, we get back to the most general case:

$$S = R \cap \mathbb{Z}^n \quad \text{where } R \text{ is a rational polyhedron}$$

and R is not necessarily pointed. Then $\mathrm{conv}(S) \cap \mathbb{Z}^n = S$ and $\mathrm{conv}(S)$ can be written as

$$\mathrm{conv}(S) = \mathcal{P} + \mathcal{R} + \mathcal{L}$$

where $\mathcal{L}$ is the lineality space of $\mathrm{conv}(S)$, $\mathcal{P} + \mathcal{R}$ is the pointed polyhedron $\mathrm{conv}(S) \cap \mathcal{L}^\perp$ whose recession cone is $\mathcal{R}$, and $\mathcal{P}$ is a polytope. Let $S_0 \subseteq \mathbb{Z}^n$ be the set of integer points such that $\mathrm{conv}(S_0) \cap \mathbb{Z}^n = S_0$ and

$$\mathrm{conv}(S_0) = \mathcal{P} + \mathrm{lin}(\mathcal{R}) + \mathcal{L}$$

where $\mathrm{lin}(\mathcal{R})$ is the linear hull of $\mathcal{R}$ or $\mathcal{R} + (-\mathcal{R})$. By definition, $S \subseteq S_0$ and $\mathrm{conv}(S_0)$ is a relaxation of $\mathrm{conv}(S)$. Moreover, $\mathrm{conv}(S_0)$ is a rational cylinder.

Lemma 6. *If $P \subseteq \mathrm{conv}(S)$ is a rational polyhedron, then*

$$P_S = P_{S_0} \cap P_{S,\Pi} \quad \textit{where} \;\; \Pi := \{(\alpha, \beta) \in \Pi_P : \; \alpha\ell = 0 \;\; \textit{for} \;\; \ell \in \mathcal{L}\}\,. \tag{13}$$

Due to the space limit, we defer the proof of Lemma 6 to a full version of this paper. By this lemma, it is sufficient to show that $P_{S,\Pi}$ is a rational polyhedron, and the following lemma will be useful for that:

Lemma 7 (Projection lemma [6]). *Let F, S and P be defined as*

$$S = F \times \mathbb{Z}^{n_2} \; \textit{for some} \; F \subseteq \mathbb{Z}^{n_1} \quad \textit{and} \quad P = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : Ax + Cy \le b\}$$

where the matrices A, C, b have integral components and $n_1, n_2, 1$ columns, respectively. Let $\Omega \subseteq \{(\alpha, \beta) \in \Pi_P : \alpha = (\phi, \mathbf{0}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\}$, and let $\Phi = \{(\phi, \beta) \in \mathbb{R}^{n_1} \times \mathbb{R} : (\phi, \mathbf{0}) = \alpha,\; (\alpha, \beta) \in \Omega\}$. If $Q = \mathrm{proj}_x(P)$, then $P_{S,\Omega} = P \cap (Q_{F,\Phi} \times \mathbb{R}^{n_2})$.

Now we are ready to prove the main result of this paper:

Proof of Theorem 1. Let Π be defined as in (13). By Lemma 6, we know that $P_S = P_{S_0} \cap P_{S,\Pi}$. Since $\mathrm{conv}(S_0)$ is a rational cylinder, Theorem 3 implies that P_{S_0} is a rational polyhedron. So, it is sufficient to show that $P_{S,\Pi}$ is a rational polyhedron. Since $\mathcal{P} + \mathcal{R} = \mathrm{conv}(S) \cap \mathcal{L}^\perp$, there exists a unimodular transformation τ such that $\tau(\mathcal{L}) = \{\mathbf{0}\} \times \mathbb{R}^{n_2}$ and $\tau(\mathcal{P} + \mathcal{R}) \subseteq \mathbb{R}^{n_1} \times \{\mathbf{0}\}$. Let $Q = \tau(P)$ and $T = \tau(S)$. Then Lemma 1 implies that

$$\tau(P_{S,\Pi}) = Q_{T,\Omega} \quad \text{where} \;\; \Omega := \{(\alpha, \beta) \in \Pi_Q : \; \alpha\ell = 0 \;\; \text{for} \;\; \ell \in \tau(\mathcal{L})\}.$$

Since $\tau(\mathcal{L}) = \{\mathbf{0}\} \times \mathbb{R}^{n_2}$, we have $\Omega = \{(\alpha, \beta) \in \Pi_Q : \; \alpha_{n_1+1} = \cdots = \alpha_{n_1+n_2} = 0\}$. Moreover, T can be written as $T = T_C \times \mathbb{Z}^{n_2}$ where $\mathrm{conv}(T_C) \subseteq \mathbb{R}^{n_1}$ is a pointed polyhedron and $T_C = \mathrm{conv}(T_C) \cap \mathbb{Z}^{n_1}$. Let

$$\Phi = \{(\phi, \beta) \in \mathbb{R}^{n_1} \times \mathbb{R} : \; (\phi, \mathbf{0}) = \alpha, \; (\alpha, \beta) \in \Omega\}\,.$$

Let $\hat{Q}$ denote the projection of Q onto the $\mathbb{R}^{n_1}$-space. As $Q \subseteq \mathrm{conv}(S)$, we have $\hat{Q} \subseteq \mathrm{conv}(T_C)$ and $\Phi = \Pi_{\hat{Q}}$. Since $\mathrm{conv}(T_C)$ is pointed, we know from Theorem 7 that $\hat{Q}_{T_C,\Phi}$ is a rational polyhedron. Since $Q_{T,\Omega} = Q \cap \left(\hat{Q}_{T_C,\Phi} \times \mathbb{R}^{n_2}\right)$ by Lemma 7, it follows that $Q_{T,\Omega}$ is a rational polyhedron. Therefore, P_S is a rational polyhedron, as required. $\square$

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A Proof of Lemma 5

First of all, Claims 1 and 2 below are proved inside the proof of Lemma 4.10 in [6].

Claim 1 *$\mu \geq \mathbf{0}$ and supp(μ) = supp(λ).*

Claim 2 *$\mu b = \min\left\{\mu A x : x \in P^{\uparrow}\right\}$ and therefore $(\mu A, \mu b) \in \Pi_{P^{\uparrow}}$.*

Since $\operatorname{supp}(\mu) = \operatorname{supp}(\lambda)$ by Claim 1 and $Ar^j \geq \mathbf{0}$ for all $j \in N_r$, it follows that r-$\operatorname{supp}(\mu A) = r$-$\operatorname{supp}(\lambda A)$, and therefore, $t(\mu, A) = t(\lambda, A)$.

The next claim extends Claim 3 of Lemma 4.10 in [6].

Claim 3 *Let $Q = \left\{x \in \operatorname{cone}\left\{e^1, \ldots, e^{n_1}, r^1, \ldots, r^{n_r}\right\} : \ \mu b \leq \mu A x \leq \mu b + \Delta\right\}$. There is no point $x \in Q$ that satisfies*

$$\sum_{i=1}^{\ell} p_i a_i x \geq 1 + \sum_{i=1}^{\ell} p_i b_i. \tag{14}$$

Proof. Suppose for a contradiction that there exists $\tilde{x} \in Q$ satisfying (14). Recall that for the index k defined in (8), the inequality $\mu A r^k > 0$ holds. Let $v = \frac{\mu b}{\mu A r^k} r^k$. Then $\mu A v = \mu b$ and $v \in Q$. In addition, for the index ℓ defined in (10), we have $\sum_{i=1}^{\ell} p_i a_i v = 0$ since $k \notin \bigcup_{i=1}^{t-1} r\text{-supp}(a_i)$ and $a_i r^k = 0$ for $i \leq t-1$. As $\tilde{x} \in Q$ satisfies (14) and $v \in Q$ satisfies $\sum_{i=1}^{\ell} p_i a_i v = 0$, we can take a convex combination of these points to get a point $\bar{x} \in Q$ such that

$$\sum_{i=1}^{\ell} p_i a_i \bar{x} = 1 + \sum_{i=1}^{\ell} p_i b_i \quad \Rightarrow \quad \sum_{i=1}^{\ell} p_i (a_i \bar{x} - b_i) = 1. \tag{15}$$

As $\mu A \bar{x} \leq \mu b + \Delta$, we have

$$\sum_{i=1}^{\ell} \mu_i (a_i \bar{x} - b_i) \leq - \sum_{j=\ell+1}^{m} \mu_j (a_j \bar{x} - b_j) + \Delta. \tag{16}$$

As in [6], we can rewrite (16) as:

$$\begin{aligned} \frac{\lambda_\ell}{p_\ell}\left(1 + \sum_{i=1}^{\ell} \varepsilon_i (a_i \bar{x} - b_i)\right) &\leq - \sum_{j=\ell+1}^{m} \mu_j (a_j \bar{x} - b_j) + 2\Delta \\ &\leq \sum_{j=\ell+1}^{m} \mu_j b_j + 2\Delta \leq \lambda_{\ell+1}(mB + 2D) = \frac{1}{2}\lambda_{\ell+1} M_1 \end{aligned} \tag{17}$$

where the second inequality in (17) follows from the assumption that $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ satisfy (3), the third inequality follows from the fact that $\mu_i = \lambda_i \leq$

$\lambda_{\ell+1}$ for $i = \ell+1, \ldots, m$ by (12) and that $b_j \leq B$ by Definition 2, and the last equality simply follows from the definition of M_1.

Next, we obtain a lower bound on the first term in (17). As $a_i\bar{x} \geq 0$, $b_i \geq 0$, and $\varepsilon_i \in [-\varepsilon, \varepsilon]$, we have

$$\sum_{i=1}^{\ell} \varepsilon_i \left(a_i\bar{x} - b_i\right) = \sum_{i=1}^{\ell} \varepsilon_i a_i \bar{x} - \sum_{i=1}^{\ell} \varepsilon_i b_i \geq -\varepsilon \sum_{i=1}^{\ell} (a_i\bar{x} + b_i). \tag{18}$$

Following the same argument in Claim 3 of Lemma 4.10 in [6], we can show that $-\varepsilon \sum_{i=1}^{\ell}(a_i\bar{x}+b_i) \geq -\frac{1}{2}$. Then it follows from (18) that $\sum_{i=1}^{\ell} \varepsilon_i(a_i\bar{x}-b_i) \geq -1/2$. So, the left hand side of (17) is lower bounded by $\lambda_\ell/2p_\ell$.

Since the first term in (17) is at least $\lambda_\ell/2p_\ell$, we obtain $\lambda_\ell \leq p_\ell\lambda_{\ell+1}M_1$ from (17), implying in turn that $M_\ell < p_\ell M_1$ as we assumed that $\lambda_\ell > M_\ell\lambda_{\ell+1}$ as in (10). However, (11) implies that $M_\ell \geq p_\ell M_1$, a contradiction. □

Claim 4 *$\mu A x \geq \lceil \mu b \rceil_{S,\mu A}$ dominates $\lambda A x \geq \lceil \lambda b \rceil_{S,\lambda A}$.*

Proof. We will first show that

$$\mu b \leq \lceil \mu b \rceil_{S,\mu A} \leq \mu b + \Delta \tag{19}$$

holds. Set $(\alpha, \beta) = (\mu A, \mu b)$. By Claim 2, we have that $\beta = \min\{\alpha x : x \in P^{\uparrow}\}$. As the extreme points of $P^{\uparrow}$ are contained in $\operatorname{conv}(S)$, it follows that $\beta \geq \min\{\alpha z : z \in S\}$. If $\beta = \min\{\alpha z : z \in S\}$, then $\beta = \lceil \beta \rceil_{S,\alpha}$. Thus we may assume that $\beta > \min\{\alpha z : z \in S\}$, so there exists $z' \in S$ such that $\beta > \alpha z'$. Remember that by (7), $\Delta = \min\{\lambda A r^j : j \in r\text{-supp}(\lambda A)\}$, and let j be such that $\lambda A r^j = \Delta$. As $r\text{-supp}(\lambda A) = r\text{-supp}(\mu A)$, we have $\alpha r^j > 0$ and $\kappa = (\beta - \alpha z')/\alpha r^j > 0$. Therefore $z'' = z' + \lceil \kappa \rceil r^j \in S$. Observe that

$$\beta = \alpha z' + (\beta - \alpha z') = \alpha\left(z' + \kappa r^j\right) \leq \alpha\left(z' + \lceil \kappa \rceil r^j\right) = \beta + \alpha r^j(\lceil \kappa \rceil - \kappa) \leq \beta + \alpha r^j.$$

As $\lambda \geq \mu$, we have $\Delta \geq \alpha r^j$ implying $\beta \leq \alpha z'' \leq \beta + \Delta$ and (19) holds, as desired.

Using (19), we will show that $\mu A x \geq \lceil \mu b \rceil_{S,\mu A}$ dominates $\lambda A x \geq \lceil \lambda b \rceil_{S,\lambda A}$. Let $z \in S$ be such that $\mu A z = \lceil \mu b \rceil_{S,\mu A}$. As z is integral and $\mu b \leq \lceil \mu b \rceil_{S,\mu A} \leq \mu b + \Delta$ by (19), Claim 3 implies that $\sum_{i=1}^{\ell} p_i a_i z < 1 + \sum_{i=1}^{\ell} p_i b_i$, and therefore, $\sum_{i=1}^{\ell} p_i a_i z = \sum_{i=1}^{\ell} p_i b_i - f$ for some integer $f \in [0, \sum_{i=1}^{\ell} p_i b_i]$. Consider $z + f r^j \in S$ and observe that

$$\begin{aligned}\lambda A\left(z + f r^j\right) = \lambda A z + f \lambda A r^j &= \left(\mu A + \Delta \sum_{i=1}^{\ell} p_i a_i\right) z + \Delta \sum_{i=1}^{\ell} p_i (b_i - a_i z) \\ &= \lceil \mu b \rceil_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i.\end{aligned}$$

Since $\lceil \mu b \rceil_{S,\mu A} \geq \mu b$, we must have $\lceil \mu b \rceil_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i \geq \mu b + \Delta \sum_{i=1}^{\ell} p_i b_i = \lambda b$. Then $\lceil \mu b \rceil_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i \geq \lceil \lambda b \rceil_{S,\lambda A}$. Then the inequality $\lambda A x \geq \lceil \lambda b \rceil_{S,\lambda A}$ is dominated by $\mu A x \geq \lceil \mu b \rceil_{S,\mu A}$, as the former is implied by the latter and a nonnegative combination of the inequalities in $Ax \geq b$, as required. □