# Intersecting restrictions in clutters

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**Abstract**

A clutter is *intersecting* if the members do not have a common element yet every two members intersect. It has been conjectured that for clutters without an intersecting minor, total primal integrality and total dual integrality of the corresponding set covering linear system must be equivalent. In this paper, we provide a polynomial characterization of clutters without an intersecting minor.

One important class of intersecting clutters comes from projective planes, namely the *deltas*, while another comes from graphs, namely the *blockers of extended odd holes*. Using similar techniques, we will provide a polynomial algorithm for finding a delta or the blocker of an extended odd hole minor in a given clutter. This result is quite surprising as the same problem is NP-hard if the input were the blocker instead of the clutter.

## 1 Introduction

All sets considered in this paper are finite. Let $V$ be a set of *elements*, and let $\mathcal{C}$ be a family of subsets of $V$ called *members*. $\mathcal{C}$ is a *clutter* over *ground set* $V$ if no member contains another [10]. The two clutters $\{\}, \{\emptyset\}$ are called *trivial* while the other ones are called *nontrivial*.

A *cover* is a subset $B \subseteq V$ such that $B \cap C \neq \emptyset$ for all $C \in \mathcal{C}$ ([20], Volume C, §77.5). The *covering number*, denoted by $\tau(\mathcal{C})$, is the minimum cardinality of a cover while the *packing number*, denoted by $\nu(\mathcal{C})$, is the maximum number of pairwise disjoint members. Notice that $\tau(\mathcal{C}) \geq \nu(\mathcal{C})$. We say that $\mathcal{C}$ is an *intersecting clutter* if $\tau(\mathcal{C}) \geq 2$ and $\nu(\mathcal{C}) = 1$, that is, if the members do not have a common element yet every two members intersect. We do not consider the trivial clutters intersecting.

Given disjoint $I, J \subseteq V$, the *minor* of $\mathcal{C}$ obtained after *deleting* $I$ and *contracting* $J$ is the clutter over ground set $V - (I \cup J)$ whose members are

$$\mathcal{C} \setminus I/J := \text{ the inclusion-wise minimal sets of } \{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}.$$

If $I \cup J \neq \emptyset$, then $\mathcal{C} \setminus I/J$ is a *proper* minor.

**Conjecture 1.1.** *Let $\mathcal{C}$ be a clutter over ground set $V$. For $w \in \mathbb{Z}_+^V$ consider the dual pair of linear programs*

$$(P) \quad \begin{array}{lll} \min & w^\top x & \\ \textit{s.t.} & \sum (x_u : u \in C) \geq 1 & \forall C \in \mathcal{C} \\ & x \geq \mathbf{0} & \end{array} \qquad (D) \quad \begin{array}{lll} \max & \mathbf{1}^\top y & \\ \textit{s.t.} & \sum (y_C : u \in C \in \mathcal{C}) \leq w_u & \forall u \in V \\ & y \geq \mathbf{0}. & \end{array}$$

*If $\mathcal{C}$ has no intersecting clutter as a minor, then the following statements are equivalent:*

*(i)* $(P)$ *has an integral optimal solution for all* $w \in \mathbb{Z}_+^V$,

*(ii)* $(D)$ *has an integral optimal solution for all* $w \in \mathbb{Z}_+^V$.

We will see in §2 that Conjecture 1.1 is a simple rephrasing of the $\tau = 2$ *Conjecture* of Cornuéjols, Guenin and Margot [7]. Upon a first reading of Conjecture 1.1, a natural question comes to mind: What does it mean for a clutter *not* to have an intersecting minor? We will provide a polynomial characterization of this property.

Let $I \subseteq V$ and let

$$J := \{u \in V - I : \{u\} \text{ is a cover of } \mathcal{C} \setminus I\}.$$

We refer to $\mathcal{C} \setminus I/J$ as a *restriction* of $\mathcal{C}$ and say that it is obtained from $\mathcal{C}$ after *restricting* $I$.[1] If $I \cup J \neq \emptyset$, then $\mathcal{C} \setminus I/J$ is a *proper* restriction. Notice that a nontrivial restriction has covering number at least two.

**Remark 1.2.** *A clutter has an intersecting minor if, and only if, it has an intersecting restriction.*

*Proof.* $(\Leftarrow)$ is immediate. $(\Rightarrow)$ Let $\mathcal{C}$ be a clutter over ground set $V$. Pick disjoint $I, J \subseteq V$ such that $\mathcal{C} \setminus I/J$ is an intersecting clutter. We will prove that restricting $I$ gives an intersecting minor as well. Let $J' := \{u \in V - I : \{u\} \text{ is a cover of } \mathcal{C} \setminus I\}$. Since $\tau(\mathcal{C} \setminus I/J) \geq 2$, $J' \subseteq J$. As a result, since $\mathcal{C} \setminus I/J$ does not have disjoint members, neither does $\mathcal{C} \setminus I/J'$. Moreover, as a nontrivial restriction of $\mathcal{C}$, $\tau(\mathcal{C} \setminus I/J') \geq 2$. Hence, $\mathcal{C} \setminus I/J'$ is an intersecting restriction. $\square$

Thus, to look for an intersecting minor in $\mathcal{C}$, we can just go through the restrictions, thereby bringing the size of the search space from $3^{|V|}$ down to $2^{|V|}$. In §3 we prove the following characterization:

**Theorem 1.3.** *Let $\mathcal{C}$ be a clutter over ground set $V$. Then the following statements are equivalent:*

*(i)* $\mathcal{C}$ *has an intersecting minor,*

*(ii) there are distinct members* $C_1, C_2, C_3$ *such that restricting* $V - (C_1 \cup C_2 \cup C_3)$ *yields no two disjoint members.*

This turns out to be a polynomial characterization due to the following immediate consequence:

**Theorem 1.4.** *Take integers $n, m \geq 1$ and a clutter over $n$ elements and $m$ members. Then one can find an intersecting minor, or certify that none exists, in time $O(nm^5)$.*

*Proof.* Let $\mathcal{C}$ be a clutter over ground set $V$ such that $|V| = n$ and $|\mathcal{C}| = m$. Consider the following algorithm:

1. For all distinct $C_1, C_2, C_3 \in \mathcal{C}$:

   (i) let $\mathcal{C}'$ be the restriction of $\mathcal{C}$ obtained after restricting $V - (C_1 \cup C_2 \cup C_3)$,

   (ii) if $\mathcal{C}'$ does not have disjoint members, then output $\mathcal{C}'$ as an intersecting minor.

[1] This definition is a slight departure from the one proposed in [20], Volume C, §77.2.

2. If (ii) fails for every triple, then there is no intersecting minor.

The correctness of this algorithm is guaranteed by Theorem 1.3. Its running time is $\binom{m}{3} \times O(nm^2) = O(nm^5)$, as required. $\square$

Hence, the hypothesis of Conjecture 1.1 can be tested in time polynomial in the size of the input. This theorem confirms Conjecture 2.14 of [1].

## 1.1 Examples of intersecting clutters

Let $\mathcal{C}$ be a clutter over ground set $V$. A cover is *minimal* if it does not contain another cover. The *blocker* of $\mathcal{C}$, denoted $b(\mathcal{C})$, is the clutter over ground set $V$ of the minimal covers of $\mathcal{C}$ [10]. It can be readily checked that $b(b(\mathcal{C})) = \mathcal{C}$ [14, 10] and $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ for disjoint $I, J \subseteq V$ [21]. Two clutters $\mathcal{C}_1, \mathcal{C}_2$ are *isomorphic* if one is obtained from the other after relabeling its ground set. Let us now provide three important examples of intersecting clutters.

Consider the clutter over ground set $\{1,2,3,4,5,6\}$ whose members are $\{1,3,5\}, \{1,4,6\}, \{2,3,6\}$ and $\{2,4,5\}$. This clutter is denoted $Q_6$ [22]. Notice that $Q_6$ is the clutter of the triangles of $K_4$, the complete graph on four vertices whose edges form the ground set of the clutter. Since $\tau(Q_6) = 2 > 1 = \nu(Q_6)$, $Q_6$ is an intersecting clutter.

Take an integer $n \geq 3$. Let $\Delta_n$ be the clutter over ground set $[n]$ whose members are $\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3,\ldots,n\}$.[2] Any clutter isomorphic to $\Delta_n$ is referred to as a *delta of dimension* $n$ [4]. Observe that the elements and members of a delta correspond to the points and lines of a degenerate projective plane. Notice that $\Delta_n = b(\Delta_n)$, and that $\Delta_n$ is an intersecting clutter.

Assume that $n$ is an odd integer and satisfies $n \geq 5$. Let $\mathcal{C}$ be a clutter over ground set $[n]$ whose minimum cardinality members are $\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}$. Notice that $\mathcal{C}$ may have members of cardinality at least three. Any clutter isomorphic to $\mathcal{C}$ is referred to as an *extended odd hole of dimension* $n$ [5]. Since the members of $\mathcal{C}$ have cardinality at least two, the minimal covers do not have a common element. Moreover, the minimum cardinality members of $\mathcal{C}$ guarantee that every minimal cover has cardinality at least $\frac{n+1}{2}$. In particular, every two minimal covers intersect. Thus, the blocker of an extended odd hole is an intersecting clutter.

As much as they may have in common, there is a point the first example and the other two disagree on.

## 1.2 Fractional packings of value two

Let $\mathcal{C}$ be a clutter over ground set $V$. A *fractional packing of value two* is a vector $y \in \mathbb{R}^{\mathcal{C}}_+$ such that

$$\sum (y_C : v \in C \in \mathcal{C}) \leq 1 \quad \forall v \in V \qquad \text{and} \qquad \mathbf{1}^\top y = 2.$$

If there are disjoint members, then there is a fractional packing of value two. The converse however does not hold:

[2] $[n] := \{1,2,\ldots,n\}$

**Remark 1.5.** *$Q_6$ has a fractional packing of value two.*

*Proof.* Recall that $Q_6$ is the clutter of the triangles of $K_4$, so a fractional packing of value two is obtained simply by assigning $\frac{1}{2}$ to each of the four triangles. □

The following is an immediate consequence of Farkas' Lemma:

**Lemma 1.6.** *Let $\mathcal{C}$ be a clutter over ground set $V$. Then exactly one of the following statements holds:*

*(1) $\mathcal{C}$ has a fractional packing of value two,*

*(2) there exists $w \in \mathbb{R}^V_+$ such that $\sum (w_u : u \in C) > \frac{\mathbf{1}^\top w}{2}$ for all $C \in \mathcal{C}$.*

*Proof.* Consider the dual pair of linear programs

$$(P)\quad \begin{array}{ll} \max & z \\ \text{s.t.} & \sum (w_u : u \in C) \geq z \quad \forall C \in \mathcal{C} \\ & \mathbf{1}^\top w = 1 \\ & w \geq \mathbf{0} \end{array} \qquad (D)\quad \begin{array}{ll} \min & t \\ \text{s.t.} & \sum (y_C : v \in C \in \mathcal{C}) \leq t \quad \forall v \in V \\ & \mathbf{1}^\top y = 1 \\ & y \geq \mathbf{0}. \end{array}$$

Notice that (2) holds if and only if the optimal value of $(P)$ is greater than $\frac{1}{2}$, while (1) holds if and only if the optimal value of $(D)$ is less than or equal to $\frac{1}{2}$. It therefore follows from strong linear programming duality that exactly one of (1) and (2) holds. □

As a consequence, and in contrast to $Q_6$,

**Remark 1.7.** *Neither a delta nor the blocker of an extended odd hole has a fractional packing of value two.*

*Proof.* Let $\mathcal{C}$ be a delta or the blocker of an extended odd hole of dimension, say, $n$. Define $w \in \mathbb{R}^n_+$ as follows: If $\mathcal{C} = \Delta_n$ then let $w = (n-2, 1, \ldots, 1)$, and if $\mathcal{C}$ is the blocker of an extended odd hole then let $w = (1, 1, \ldots, 1)$. Notice that $\sum (w_u : u \in C) > \frac{\mathbf{1}^\top w}{2}$ for all $C \in \mathcal{C}$. It therefore follows from Lemma 1.6 that $\mathcal{C}$ does not have a fractional packing of value two. □

## 1.3 Finding a delta or the blocker of an extended odd hole minor

Take a clutter whose covering number is at least two. If there is no fractional packing of value two, then there is a certificate given by Lemma 1.6, and using this certificate, Abdi and Lee [5] obtained a delta or the blocker of an extended odd hole minor:

**Theorem 1.8** ([5]). *Let $\mathcal{C}$ be a clutter over ground set $V$ such that $\tau(\mathcal{C}) \geq 2$. Assume that there exists $w \in \mathbb{R}^V_+$ such that $\sum (w_u : u \in C) > \frac{\mathbf{1}^\top w}{2}$ for all $C \in \mathcal{C}$. Then $\mathcal{C}$ has a delta or the blocker of an extended odd hole minor, which can be found in strongly polynomial time with running time $O(|V||\mathcal{C}| + |V|^4)$.*

Using this theorem, and deploying techniques similar to those used to prove Theorem 1.3, we prove in §4 the following fractional analogue:

**Theorem 1.9.** *Let $\mathcal{C}$ be a clutter over ground set $V$. Then the following statements are equivalent:*

*(i) $\mathcal{C}$ has a delta or the blocker of an extended odd hole minor,*

*(ii) there are distinct members $C_1, C_2, C_3$ such that restricting $V - (C_1 \cup C_2 \cup C_3)$ yields no fractional packing of value two.*

Take integers $n, m \geq 1$ and a clutter $\mathcal{C}$ over at most $n$ elements and $m$ members. Denote by $T(n, m)$ the minimum time it takes to solve a linear program of the form

$$\begin{array}{ll} \max & z \\ \text{s.t.} & \sum (w_u : u \in C) \geq z \quad \forall C \in \mathcal{C} \\ & \mathbf{1}^\top w = 1 \\ & w \geq \mathbf{0}. \end{array}$$

Keeping Lemma 1.6 in mind, notice the following immediate remark:

**Remark 1.10.** *Let $\mathcal{C}$ be a clutter over ground set $V$ such that $\tau(\mathcal{C}) \geq 2$. In time $T(|V|, |\mathcal{C}|)$, one can output a $w \in \mathbb{R}_+^V$ such that $\sum (w_u : u \in C) > \frac{\mathbf{1}^\top w}{2}$ for all $C \in \mathcal{C}$, or declare that $\mathcal{C}$ has a fractional packing of value two.*

So, what is $T(n, m)$? We know from classic linear programming results that $T(n, m)$ is bounded from above by a polynomial in $n, m$. Let us explicitly compute one such bound, say Renegar's [18]. After bringing our linear program to his standard form $\max\{c^\top x : Ax \geq b\}$, where $A$ is an $m' \times n'$ matrix and $L$ is the total number of bits needed to represent all the entries of $A, b, c$, our linear program can be solved with $O((n' + m')^{1.5} n'^2 L)$ arithmetic operations and $O((n' + m')^{1.5} n'^2 L^2 (\log L)(\log \log L))$ bit operations, the latter dominating the total running time. In our case, the reader can check that

$$m' \leq n + m + 2 \quad \text{and} \quad n' \leq n + 1 \quad \text{and} \quad L \leq (n + m + 2)(n + 1) + (n + m + 2) + (n + 1),$$

so

$$T(n, m) = O\big((n + m)^{3.5} n^4 \log(n + m) \log\log(n + m)\big).$$

We are now ready to prove the following:

**Theorem 1.11.** *Take integers $n, m \geq 1$ and a clutter over $n$ elements and $m$ members. Then one can find a delta or the blocker of an extended odd hole minor, or certify that none exists, in time at most*

$$T(n, m) \cdot m^3 + O(nm^4 + n^4 m^3).$$

*More explicitly, the running time is*

$$O\big(n^4 m^3 (n + m)^{3.5} \log(n + m) \log\log(n + m)\big).$$

*Proof.* Let $\mathcal{C}$ be a clutter over ground set $V$ such that $|V| = n$ and $|\mathcal{C}| = m$. Consider the following algorithm:

1. For all $C_1, C_2, C_3 \in \mathcal{C}$:

   (i) let $\mathcal{C}'$ be the restriction of $\mathcal{C}$ obtained after restricting $V - (C_1 \cup C_2 \cup C_3)$, and let $U$ denote the ground set of $\mathcal{C}'$,

   (ii) if there is a $w \in \mathbb{R}^U_+$ such that $\sum(w_u : u \in C) > \frac{\mathbf{1}^\top w}{2}$ for all $C \in \mathcal{C}'$, then output a delta or the blocker of an extended odd hole minor.

2. If (ii) fails for every triple, then there is no delta or the blocker of an extended odd hole minor.

The correctness of this algorithm is guaranteed by Theorems 1.8 and 1.9. Its running time by Theorem 1.8 and Remark 1.10 is

$$\binom{m}{3} \times \big(T(n,m) + O(nm + n^4)\big) \leq T(n,m) \cdot m^3 + O(nm^4 + n^4 m^3),$$

as required. $\square$

Theorem 1.11, while confirming Conjecture 2.13 of [1], is a surprising result, as the same problem would be NP-hard if the input were the blocker rather than the clutter, as was shown by Ding, Feng and Zang [9]. We will discuss this anomaly and a potential implication in §5.

## 2 The $\tau = 2$ Conjecture

Let $\mathcal{C}$ be a clutter over ground set $V$. $\mathcal{C}$ *packs* if $\tau(\mathcal{C}) = \nu(\mathcal{C})$, that is, if the minimum cardinality of a cover equals the maximum number of pairwise disjoint members [22]. $\mathcal{C}$ has the *packing property* if every minor, including the clutter itself, packs [7]. The following is a consequence of a seminal result of Lehman [16]:

**Theorem 2.1** ([7])**.** *If a clutter has the packing property, then it is ideal.*

Here, $\mathcal{C}$ is *ideal* if the linear program

$$(P) \quad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & \sum(x_u : u \in C) \geq 1 \quad \forall C \in \mathcal{C} \\ & x \geq \mathbf{0} \end{array}$$

has an integral optimal solution for all $w \in \mathbb{Z}^V_+$ [8]. If a clutter is ideal, then so is every minor of it [22].

**The Replication Conjecture** ([6])**.** *The packing property implies the max-flow min-cut property.*

Here, $\mathcal{C}$ has the *max-flow min-cut property* if the dual linear program of $(P)$,

$$(D) \quad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum(y_C : u \in C \in \mathcal{C}) \leq w_u \quad \forall u \in V \\ & y \geq \mathbf{0} \end{array}$$

has an integral optimal solution for all $w \in \mathbb{Z}_+^V$ [6]. If a clutter has the max-flow min-cut property, then so does every minor of it [22]. A classic result of Hoffman [13] and Edmonds and Giles [11] tells us that the max-flow min-cut property implies idealness. Thus the Replication Conjecture, if true, would be a strengthening of Theorem 2.1. It can be readily checked that the max-flow min-cut property implies the packing property, so the Replication Conjecture, if true, would imply that these two properties are equivalent.

In an attempt to prove the Replication Conjecture, Conuéjols, Guenin and Margot [7] made a stronger conjecture, which is equivalent to Conjecture 1.1. To elaborate, a clutter is *minimally non-packing* if it does not pack but every proper minor does. Theorem 2.1 tells us that every minimally non-packing clutter is either ideal or *minimally non-ideal* – it is not ideal but every proper minor is. The latter class is relatively well-understood thanks to the seminal result of Lehman [16]. What about the former class? All known examples of ideal minimally non-packing clutters have covering number two, so it has been conjectured that,

**The $\tau = 2$ Conjecture** ([7]). *Every ideal minimally non-packing clutter has covering number two.*

These clutters are of particular relevance to us:

**Remark 2.2.** *Let $\mathcal{C}$ be an ideal minimally non-packing clutter such that $\tau(\mathcal{C}) = 2$. Then $\mathcal{C}$ is an intersecting clutter that has a fractional packing of value two.*

*Proof.* Since $\mathcal{C}$ does not pack, $1 \leq \nu(\mathcal{C}) < \tau(\mathcal{C}) = 2$, so $\mathcal{C}$ is an intersecting clutter. Since $\mathcal{C}$ is an ideal clutter, the common optimal value of the dual linear programs

$$\begin{array}{lll} \min & \mathbf{1}^\top x & \\ \text{s.t.} & \sum (x_u : u \in C) \geq 1 & C \in \mathcal{C} \\ & x \geq \mathbf{0} & \end{array} \qquad \begin{array}{lll} \max & \mathbf{1}^\top y & \\ \text{s.t.} & \sum (y_C : u \in C \in \mathcal{C}) \leq 1 & u \in V \\ & y \geq \mathbf{0} & \end{array}$$

is two, implying in turn that $\mathcal{C}$ has a fractional packing of value two. $\square$

For instance, $Q_6$, the clutter of the triangles of $K_4$, is an ideal minimally non-packing clutter with covering number two [17, 22]. Therefore, Remark 2.2 provides another justification for why Remark 1.5 holds.

Let us say a few words on what is known about ideal minimally non-packing clutters. These clutters can be grouped into "chains" of clutters with the same covering number, and for those chains with covering number two, the clutter that has the smallest number of elements (and resides at the bottom of the chain) is essentially what is called a "cuboid" [2].[3] Apart from roughly a dozen small instances [19, 7], all known ideal minimally non-packing clutters – an infinite class [7] as well as over 700 small instances [3] – are in fact cuboids.

**Proposition 2.3** ([7]). *If the $\tau = 2$ Conjecture is true, then so is the Replication Conjecture.*

Equipped with this, we are ready to see that the $\tau = 2$ Conjecture is equivalent to Conjecture 1.1, predicting that if a clutter has no intersecting minor, then it is ideal if and only if it has the max-flow min-cut property:

**Proposition 2.4.** *The $\tau = 2$ Conjecture is equivalent to Conjecture 1.1.*

[3]These clutters have a cuboid "core", to be accurate.

*Proof.* Assume first that the $\tau = 2$ Conjecture is false. Then there is an ideal minimally non-packing clutter $\mathcal{C}$ such that $\tau(\mathcal{C}) \geq 3$. Since every proper minor of it packs, $\mathcal{C}$ has no intersecting proper minor. Pick an arbitrary element $v$. Then $\tau(\mathcal{C} \setminus v) \geq \tau(\mathcal{C}) - 1 \geq 2$. As a proper minor of $\mathcal{C}$, $\mathcal{C} \setminus v$ must pack so it has disjoint members, implying in turn that $\mathcal{C}$ has disjoint members. Thus, $\mathcal{C}$ has no intersecting minor. Since $\mathcal{C}$ does not pack, it does not have the max-flow min-cut property, so as an ideal clutter, $\mathcal{C}$ constitutes a counterexample to Conjecture 1.1. Thus, Conjecture 1.1 is false.

Assume conversely that the $\tau = 2$ Conjecture is true. Let $\mathcal{C}$ be a clutter over ground set $V$ without an intersecting minor. The max-flow min-cut property in general implies idealness, so we need to prove that the idealness of $\mathcal{C}$ implies its max-flow min-cut property. To this end, assume that $\mathcal{C}$ is ideal. Then since $\mathcal{C}$ has no intersecting minor, and the $\tau = 2$ Conjecture is true, $\mathcal{C}$ cannot have a minimally non-packing minor, implying in turn that it has the packing property. By Proposition 2.3, the Replication Conjecture is true, so $\mathcal{C}$ also has the max-flow min-cut property. Thus, Conjecture 1.1 is true. $\square$

## 3 Proof of Theorem 1.3

We will need the following tool for finding delta minors:

**Theorem 3.1** ([2], Theorem 2.1)**.** *Let $\mathcal{C}$ be a clutter. If there are distinct members of the form $\{u,v\},\{u,w\},C$ such that $C \cap \{u,v,w\} = \{v,w\}$, then $\mathcal{C}$ has a delta minor.*

A clutter is *strictly intersecting* if it is intersecting but no proper restriction is. Notice that if a clutter has an intersecting restriction, then it has a strictly intersecting restriction.

**Remark 3.2.** *Let $\mathcal{C}$ be a strictly intersecting clutter over ground set $V$. Then every intersecting minor of $\mathcal{C}$ is a contraction minor.*

*Proof.* Choose disjoint $I, J \subseteq V$ such that $\mathcal{C} \setminus I/J$ is intersecting. In particular, $\mathcal{C} \setminus I$ has an intersecting minor, so by Remark 1.2, $\mathcal{C} \setminus I$ has an intersecting restriction. Since $\mathcal{C}$ has no intersecting proper restriction, it follows that $I = \emptyset$, as required. $\square$

The following proposition is the key to proving Theorem 1.3:

**Proposition 3.3.** *A strictly intersecting clutter has three members whose union is the ground set.*

*Proof.* Let $\mathcal{C}$ be a strictly intersecting clutter over ground set $V$.

**Claim 1.** *If $\mathcal{C}$ has a delta minor, then there are three members whose union is $V$.*

*Proof of Claim.* Suppose that $\mathcal{C} \setminus I/J = \Delta_n = \{\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}\}$ for some disjoint $I, J \subseteq V$ and some integer $n \geq 3$. Since $\Delta_n$ is intersecting, it follows from Remark 3.2 that $I = \emptyset$. Consider the three members $\{1,2\},\{1,3\},\{2,3,\ldots,n\}$ of $\Delta_n$. Pick members $C_1, C_2, C_3$ of $\mathcal{C}$ such that $\{1,2\} \subseteq C_1 \subseteq \{1,2\} \cup J$, $\{1,3\} \subseteq C_2 \subseteq \{1,3\} \cup J$ and $\{2,3,\ldots,n\} \subseteq C_3 \subseteq \{2,3,\ldots,n\} \cup J$. We claim that $C_1 \cup C_2 \cup C_3 =$

$V$. Suppose otherwise. Pick an element $u \in V - (C_1 \cup C_2 \cup C_3)$. Clearly $u \in J$. Consider the minor $\mathcal{C}' := \mathcal{C} \setminus u/(J - \{u\})$ over ground set $[n]$. Then $\{1,2\}, \{1,3\}, \{2,3,\ldots,n\}$ are still members of $\mathcal{C}'$, implying in turn that $\mathcal{C}'$ is an intersecting clutter. This contradicts Remark 3.2. Thus, $C_1 \cup C_2 \cup C_3 = V$. ◊

We may therefore assume that $\mathcal{C}$ has no delta minor. As $\mathcal{C}$ is an intersecting clutter, $\tau(\mathcal{C}) \geq 2$. In fact,

**Claim 2.** *$\tau(\mathcal{C}) = 2$ and every element appears in a minimum cover.*

*Proof of Claim.* Let $u \in V$. It suffices to show that $u$ appears in a cover of cardinality two. For if not, then $\mathcal{C} \setminus u$ is a proper intersecting restriction, a contradiction as $\mathcal{C}$ is strictly intersecting. ◊

Take an element $u \in V$ and let $U := \{v \in V : \{u, v\}$ is a minimal cover of $\mathcal{C}\}$. By Claim 2, $U \neq \emptyset$.

**Claim 3.** *$U$ is not a cover.*

*Proof of Claim.* Suppose otherwise. Let $B$ be a minimal cover contained in $U$. Clearly $|B| \geq 2$. Pick distinct elements $v, w \in B$. Then by Theorem 3.1, the three minimal covers $\{u,v\}, \{u,w\}, B$ imply that $b(\mathcal{C})$ has a delta minor, implying in turn that $\mathcal{C}$ has a delta minor, contrary to our assumption. ◊

Thus there is a member $C \in \mathcal{C}$ such that $C \cap U = \emptyset$. Note that $u \in C$.

**Claim 4.** *If a member excludes $u$, then it properly contains $U$.*

*Proof of Claim.* Take a member $C'$ such that $u \notin C'$. Our definition of $U$ implies that $U \subseteq C'$. Since $\mathcal{C}$ is intersecting, $C' \cap C \neq \emptyset$, implying in turn that $C' \neq U$, as required. ◊

In fact,

**Claim 5.** *There are distinct members $C_1, C_2$ such that $u \notin C_1 \cup C_2$ and $C_1 \cap C_2 = U$.*

*Proof of Claim.* Since $\mathcal{C}$ is strictly intersecting, the restriction $\mathcal{C} \setminus u/U$ is not intersecting. That is, the restriction $\mathcal{C} \setminus u/U$ is either trivial or has disjoint members. Claim 4 however implies that this restriction is a nontrivial clutter, so $\mathcal{C} \setminus u/U$ must have a pair of disjoint members; any such pair corresponds to a pair of distinct members $C_1, C_2$ of $\mathcal{C}$ such that $u \notin C_1 \cup C_2$ and $C_1 \cap C_2 = U$, as required. ◊

**Claim 6.** $C \cup C_1 \cup C_2 = V$.

*Proof of Claim.* Suppose for a contradiction that there is an element $v \notin C \cup C_1 \cup C_2$. By Claim 2, there is an element $v'$ such that $\{v, v'\}$ is a cover of $\mathcal{C}$. Since $\{v, v'\} \cap C \neq \emptyset$, we have $v' \in C$, so $\{v, v'\} \cap U = \emptyset$, implying in turn that $\{v, v'\} \cap C_1$ or $\{v, v'\} \cap C_2$ is empty, a contradiction. ◊

This finishes the proof of the proposition. □

We are now ready to prove Theorem 1.3, claiming that for a clutter $\mathcal{C}$ over ground set $V$, the following statements are equivalent:

(i) $\mathcal{C}$ has an intersecting minor,

(ii) there are distinct members $C_1, C_2, C_3$ such that restricting $V - (C_1 \cup C_2 \cup C_3)$ yields no two disjoint members.

*Proof of Theorem 1.3.* **(ii)** $\Rightarrow$ **(i)** is immediate. **(i)** $\Rightarrow$ **(ii)**: It follows from Remark 1.2 that $\mathcal{C}$ has an intersecting restriction, implying in turn that $\mathcal{C}$ has a strictly intersecting restriction obtained after restricting, say, $I \subseteq V$. That is, for $J := \{u \in V - I : \{u\}$ is a cover of $\mathcal{C} \setminus I\}$, the minor $\mathcal{C} \setminus I/J$ is strictly intersecting and therefore has no two disjoint members. By Proposition 3.3, $\mathcal{C} \setminus I/J$ has members $C_1', C_2', C_3'$ such that $C_1' \cup C_2' \cup C_3' = V - (I \cup J)$. For each $i \in [3]$, let $C_i := C_i' \cup J$. Notice that $C_1, C_2, C_3$ are members of $\mathcal{C}$ that satisfy $I = V - (C_1 \cup C_2 \cup C_3)$. As a result, $C_1, C_2, C_3$ are the desired members. $\square$

This theorem was already proved for cuboids ([3], Theorem 1.12). Unlike the situation here, however, deltas and delta minors did not play a role in proving that special case.

# 4 Proof of Theorem 1.9

We will need the following tool:

**Theorem 4.1** ([5]). *Let $V$ be a set of cardinality at least 4. Let $\mathcal{C}$ be a clutter over ground set $V$ where $\min\{|C| : C \in \mathcal{C}\} = 2$ and the minimum cardinality members correspond to the edges of a connected bipartite graph $G$ over vertex set $V$ with bipartition $R \cup B = V$. If $R$ contains a member, then $\mathcal{C}$ has a delta or an extended odd hole minor.*

We are now ready to prove the following:

**Proposition 4.2.** *Take an odd integer $n \geq 5$, and let $\mathcal{C}$ be an extended odd hole over ground set $[n]$ whose minimum cardinality members are $\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}$. If $\mathcal{C}$ has no delta or extended odd hole proper minor, then for each $i \in [n]$,*

$$\left\{ i + 2k - 1 \mod n : k = 1, 2, \ldots, \frac{n+1}{2} \right\}$$

*is a minimal cover.*

*Proof.* It suffices to show that this set is a cover, as minimality follows from the existence of the minimum cardinality members. We may assume that $i = 1$. Suppose for a contradiction that $\{1, 2, 4, 6, \ldots, n-3, n-1\}$ is not a cover. Then there is a $C \in \mathcal{C}$ such that $C \subseteq \{3, 5, 7, \ldots, n-2, n\}$. Consider the graph $G$ over vertex set $\{2, 3, \ldots, n\}$ whose edges are $\{2,3\}, \{3,4\}, \ldots, \{n-2, n-1\}, \{n-1, n\}$. Then $G$ is a connected bipartite graph whose color classes are $B := \{2, 4, 6, \ldots, n-1\}$ and $R := \{3, 5, 7, \ldots, n-2, n\}$. Observe that the edges of $G$ correspond to the members of $\mathcal{C} \setminus 1$ of minimum cardinality. Since $C \in \mathcal{C} \setminus 1$ and $C \subseteq R$, it follows from Theorem 4.1 that $\mathcal{C} \setminus 1$ has a delta or an extended odd hole minor, a contradiction to our assumption. $\square$

Let $\mathcal{C}$ be a clutter over ground set $V$ such that $\tau(\mathcal{C}) \geq 2$. We say that $\mathcal{C}$ is *dense* if there exists $w \in \mathbb{R}_+^V$ such that $\sum(w_u : u \in C) > \frac{\mathbf{1}^\top w}{2}$ for all $C \in \mathcal{C}$. Recall that by Lemma 1.6, $\mathcal{C}$ is dense or has a fractional packing of value two, but not both. As was shown in Remark 1.7, deltas and blockers of extended odd holes are dense, and in fact, by Theorem 1.8, every dense clutter has a delta or the blocker of an extended odd hole minor.

**Remark 4.3.** *Let $\mathcal{C}$ be a clutter over ground set $V$ such that $\tau(\mathcal{C}) \geq 2$. If a contraction minor of $\mathcal{C}$ is dense, then so is $\mathcal{C}$.*

*Proof.* Assume that $\mathcal{C}/J$ is dense for some $J \subseteq V$. Then there is a $w \in \mathbb{R}_+^{V-J}$ such that $\sum(w_u : u \in C') > \frac{\mathbf{1}^\top w}{2}$ for all $C' \in \mathcal{C}/J$. Extend $w$ to a vector in $\mathbb{R}_+^V$ by setting $w_e := 0$ for all $e \in J$. It can be readily checked that $\sum(w_u : u \in C) > \frac{\mathbf{1}^\top w}{2}$ for all $C \in \mathcal{C}$, implying in turn that $\mathcal{C}$ is dense. $\square$

As a consequence,

**Remark 4.4.** *Let $\mathcal{C}$ be a clutter over ground set $V$. Then the following statements are equivalent:*

*(i) $\mathcal{C}$ has a delta or the blocker of an extended odd hole minor,*

*(ii) $\mathcal{C}$ has a dense restriction.*

*Proof.* **(ii)** $\Rightarrow$ **(i)** follows immediately from Theorem 1.8. **(i)** $\Rightarrow$ **(ii)**: Assume that $\mathcal{C} \setminus I/J$ is a delta or the blocker of an extended odd hole for some disjoint $I, J \subseteq V$. Then $\mathcal{C} \setminus I/J$ is dense. Let $J' := \{u \in V - I : \{u\} \text{ is a cover of } \mathcal{C} \setminus I\}$. Then $J' \subseteq J$, so by Remark 4.3, the restriction $\mathcal{C} \setminus I/J'$ is dense, as required. $\square$

A clutter is *strictly dense* if it is dense but no proper restriction is. Notice that if a clutter has a dense restriction, then it has a strictly dense restriction. The following is the key to proving Theorem 1.9:

**Proposition 4.5.** *A strictly dense clutter has three members whose union is the ground set.*

*Proof.* Let $\mathcal{C}$ be a strictly dense clutter over ground set $V$.

**Claim 1.** *No proper deletion minor of $\mathcal{C}$ has a delta or the blocker of an extended odd hole minor.*

*Proof of Claim.* If so, then by Remark 4.4, a proper deletion minor of $\mathcal{C}$ has a dense restriction, implying in turn that a proper restriction of $\mathcal{C}$ is dense, a contradiction as $\mathcal{C}$ is strictly dense. $\Diamond$

By Theorem 1.8, $\mathcal{C}$ has a delta or the blocker of an extended odd hole minor, and any such minor must be a contraction minor by Claim 1. Pick a maximal $J \subseteq V$ such that $\mathcal{C}/J$ is a delta or the blocker of an extended odd hole. Our maximal choice of $J$ implies that every proper minor of $\mathcal{C}/J$ is different from a delta or the blocker of an extended odd hole.

**Claim 2.** *$\mathcal{C}/J$ has members $C_1', C_2', C_3'$ such that $C_1' \cap C_2' \cap C_3' = \emptyset$ and $C_1' \cup C_2' \cup C_3' = V - J$.*

*Proof of Claim.* If $\mathcal{C}/J = \Delta_n$, then let $C_1' := \{1,2\}, C_2' := \{1,3\}$ and $C_3' := \{2,3,\ldots,n\}$. Otherwise, $\mathcal{C}/J$ is the blocker of an extended odd hole. We may assume that $\mathcal{C}/J$ has ground set $\{1,2,\ldots,n\}$ and its minimum cardinality covers are $\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}$. Since every proper minor of $\mathcal{C}/J$ is different from a delta or the blocker of an extended odd hole, we may apply Proposition 4.2 to get that for $i \in [3]$,

$$C_i' := \left\{ i + 2k - 1 \mod n : k = 1, 2, \ldots, \frac{n+1}{2} \right\}$$

is a member of $\mathcal{C}/J$. In both cases, it can be readily checked that $C_1', C_2', C_3'$ are the desired members. $\Diamond$

For each $i \in [3]$, pick a member $C_i$ of $\mathcal{C}$ such that $C_i' \subseteq C_i \subseteq C_i' \cup J$.

**Claim 3.** $C_1 \cup C_2 \cup C_3 = V$.

*Proof of Claim.* Suppose for a contradiction that there is an element $u \in V - (C_1 \cup C_2 \cup C_3)$. Since $C_1' \cup C_2' \cup C_3' = V - J$, $u \in J$. Consider the minor $\mathcal{C}' := \mathcal{C} \setminus u/(J - \{u\})$. As $C_1', C_2', C_3'$ are members of $\mathcal{C}/J$, they are still members of $\mathcal{C}'$. Since $C_1' \cap C_2' \cap C_3' = \emptyset$, it follows that $\tau(\mathcal{C}') \geq 2$. Since $\mathcal{C}/J$ is a delta or the blocker of an extended odd hole, it is a dense clutter. Since every member of $\mathcal{C}'$ contains a member of $\mathcal{C}/J$, and the two clutters have the same ground set, $\mathcal{C}'$ must be dense too. Thus by Theorem 1.8, $\mathcal{C}'$ and therefore $\mathcal{C} \setminus u$ has a delta or the blocker of an extended odd hole minor, a contradiction to Claim 1. $\Diamond$

This claim finishes the proof of the proposition. $\square$

We are now ready to prove Theorem 1.9:

*Proof of Theorem 1.9.* Let $\mathcal{C}$ be a clutter over ground set $V$. By Lemma 1.6, we need to show that the following statements are equivalent:

(i) $\mathcal{C}$ has a delta or the blocker of an extended odd hole minor,

(ii) there are distinct members $C_1, C_2, C_3$ such that restricting $V - (C_1 \cup C_2 \cup C_3)$ yields a dense clutter.

**(ii)** $\Rightarrow$ **(i)** follows from Theorem 1.8. **(i)** $\Rightarrow$ **(ii)**: By Remark 4.4, $\mathcal{C}$ has a dense restriction, implying in turn that $\mathcal{C}$ has a strictly dense restriction obtained after restricting, say, $I \subseteq V$. That is, for $J := \{u \in V - I : \{u\} \text{ is a cover of } \mathcal{C} \setminus I\}$, the minor $\mathcal{C} \setminus I/J$ is strictly dense. By Proposition 4.5, $\mathcal{C} \setminus I/J$ has members $C_1', C_2', C_3'$ such that $C_1' \cup C_2' \cup C_3' = V - (I \cup J)$. For each $i \in [3]$, let $C_i := C_i' \cup J$. Notice that $C_1, C_2, C_3$ are members of $\mathcal{C}$ that satisfy $I = V - (C_1 \cup C_2 \cup C_3)$. As a result, $C_1, C_2, C_3$ are the desired members. $\square$

# 5 Discussion on complexity and another conjecture

Recall that a clutter is minimally non-ideal if it is not ideal but every proper minor is. Take an odd integer $n \geq 5$. An *odd hole of dimension* $n$ is an extended odd hole of dimension $n$ without a member of cardinality at least three. Deltas, odd holes, as well as some other extended odd holes, are minimally non-ideal [15, 22].

**Theorem 5.1** ([16])**.** *The* 3*-dimensional delta, and odd holes are the only minimally non-ideal clutters where every element appears in at most two members.*

As an immediate consequence,

**Corollary 5.2.** *Let $\mathcal{C}$ be a clutter where every element appears in exactly two members. Then the following statements are equivalent:*

*(i) $\mathcal{C}$ is ideal,*

*(ii) $\mathcal{C}$ has no $\Delta_3$ or odd hole minor,*

*(iii) $\mathcal{C}$ has no delta or extended odd hole minor.*

Ding, Feng and Zang showed the following surprising result:

**Theorem 5.3** ([9], Theorem 1.5 (1))**.** *Let $\mathcal{C}$ be a clutter where every element appears in exactly two members. Then the problem "Is $\mathcal{C}$ an ideal clutter?" is co-NP-complete.*

Corollary 5.2 and Theorem 5.3 have the following immediate consequence:

**Corollary 5.4.** *Let $\mathcal{C}$ be a clutter over ground set $V$ where every element belongs to exactly two members. Then the following problems are NP-complete:*

*(i) Does $\mathcal{C}$ have a $\Delta_3$ or an odd hole minor?*

*(ii) Does $\mathcal{C}$ have a delta or an extended odd hole minor?*

*(iii) Does $\mathcal{C}$ have an odd hole minor?*

*(iv) Does $\mathcal{C}$ have an extended odd hole minor?*

*Proof.* These four problems are clearly in NP. Corollary 5.2 and Theorem 5.3 imply that **(i)** and **(ii)** are NP-complete. It can be readily checked that the problem "Does $\mathcal{C}$ have a $\Delta_3$ minor?" belongs to P. Thus **(iii)** is also NP-complete. Clearly, $\mathcal{C}$ has a delta minor if and only if it has a $\Delta_3$ minor. Thus, the problem "Does $\mathcal{C}$ have a delta minor?" belongs to P also. So **(iv)** is NP-complete as well. □

Given a clutter $\mathcal{C}$, notice that finding a delta or an extended odd hole minor in $\mathcal{C}$ is mathematically, but not computationally, equivalent to finding a delta or the blocker of an extended odd hole minor in $b(\mathcal{C})$. As a consequence, Corollary 5.4 (ii) is in total contrast with Theorem 1.11. These results are not at odds with each other because $\mathcal{C}$ and $b(\mathcal{C})$ may have different complexity. Nevertheless,

**Theorem 5.5** ([15])**.** *A clutter is ideal if and only if its blocker is ideal.*

Theorems 1.11 and 5.5 suggest the following:

**Conjecture 5.6.** *There is an algorithm that given clutters $\mathcal{C}, \mathcal{B}$ over ground set $V$ outputs one of the following in time polynomial in $|V|, |\mathcal{C}|, |\mathcal{B}|$:*

*(i)* *$\mathcal{C}, \mathcal{B}$ are not blockers,*

*(ii)* *at least one of $\mathcal{C}, \mathcal{B}$ is not ideal,*

*(iii)* *$\mathcal{C}, \mathcal{B}$ are blocking ideal clutters.*

Determining whether or not two clutters are blockers is computationally and mathematically equivalent to determining whether or not two monotone disjunctive normal forms are duals, and as such we have the following result by Fredman and Khachiyan:

**Theorem 5.7** ([12])**.** *Let $\mathcal{C}, \mathcal{B}$ be clutters over the same ground set, and let $n := |\mathcal{C}| + |\mathcal{B}|$. Then in time $n^{O(\log n)}$, one can determine whether or not $\mathcal{C}, \mathcal{B}$ are blockers.*

It was pointed out in the proof of Corollary 5.4 that finding a $\Delta_3$ minor belongs to P. In fact, finding a delta minor in general belongs to P:

**Theorem 5.8** ([2], Theorem 2.3)**.** *There is an algorithm that given a clutter $\mathcal{C}$ over ground set $V$ finds a delta minor, or certifies that none exists, in time polynomial in $|V|$ and $|\mathcal{C}|$.*

Theorems 1.11 and 5.8 lead to the following natural question:

**Question 5.9.** *Given a clutter $\mathcal{C}$ over ground set $V$, what is the complexity of finding the blocker of an extended odd hole minor in $\mathcal{C}$?*


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