Cutting Planes and Integrality of Polyhedra: Structure and Complexity

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Integer linear programming (ILP)

Integer linear programming is an optimization problem of the following form:

$$\min\left\{c^\top x : \ Ax \geq b, \ x \in \mathbb{Z}^n\right\} \tag{ILP}$$

where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$

- If the LP relaxation, $\min\left\{c^\top x : \ Ax \geq b, \ x \in \mathbb{R}^n\right\}$, has an integral optimal solution, then it is an optimal solution to (ILP).
- If the polyhedron $\{x \in \mathbb{R}^n : Ax \geq b\}$ is integral, then there is an integral optimal solution to the LP relaxation.
- If not, we use cutting-plane methods in combination with enumeration (branch-and-bound) in practice.

Part I: Cutting planes for integer programming

- Chapter 2: Polytopes with Chvátal rank 1
- Chapter 3: Polytopes with split rank 1
- Chapter 4: Polytopes in the 0,1 hypercube that have a small Chvátal rank
- Chapter 5: Generalized Chvátal closure

Part II: Integrality of set covering polyhedra

- Chapter 6: Intersecting restrictions in clutters
- Chapter 7: Multipartite clutters
- Chapter 8: The reflective product
- Chapter 9: Ideal vector spaces

Part I (Chapters 2 – 5): Cutting planes for integer programming

Based on

(1) On the rational polytopes with Chvátal rank 1 with G. Cornuéjols and Y. Li, *Math. Program. A*, in press.
(2) On the NP-hardness of deciding emptiness of the split closure of a rational polytope in the 0,1 hypercube, *Discrete Optimization*, in press.
(3) On some polytopes contained in the 0,1 hypercube that have a small Chvatal rank with G. Cornuéjols, *Math. Program. B*, 2018.
(4) Generalized Chvátal-Gomory closures for integer programs with bounds on variables with S. Dash and O. Günlük, to be submitted.

- The Chvátal closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : \ Ax \geq b\}$ is defined as

$$P' := \bigcap_{c \in \mathbb{Z}^n} \left\{ x \in \mathbb{R}^n : \quad \underbrace{cx \geq \left\lceil \min_{y \in P} cy \right\rceil}_{\text{the Chvátal-Gomory cut}} \right\}$$

Theorem [Chvátal, 1973, Schrijver, 1980]

Let P be a rational polyhedron, and let $P_I := conv(P \cap \mathbb{Z}^n)$. Then
(1) P' is also a rational polyhedron,
(2) there exists a positive integer k such that $P^{(k)} = P_I$.

- The kth Chvátal closure of P is defined as

$$P^{(k)} := \underbrace{((P')' \cdots)'}_{k}$$

- The Chvátal rank of P is the smallest integer k such that $P^{(k)} = P_I$.

- Bounds on the Chvátal rank of a polytope in the 0,1 hypercube:

Theorem [Eisenbrand and Schulz, 2003]

Let $P \subseteq [0,1]^n$ be a polytope. Then the Chvátal rank of P is $O(n^2 \log n)$.

Theorem [Rothvoß and Sanità, 2013]

There exists a polytope $P \subseteq [0,1]^n$ whose Chvátal rank is $\Omega(n^2)$.

- When does a polytope in the 0,1 hypercube have a small Chvátal rank?

Theorem [Cornuéjols and Lee, 2018] (in Chapter 4)

Let $P \subseteq [0,1]^n$ be a polytope, and let G_n denote the skeleton graph of $[0,1]^n$. Let $\bar{S} := \{0,1\}^n \setminus P$. Then the following statements hold:

1. *if $\bar{S}$ is a stable set in G_n, then the Chvátal rank of P is at most 1,*
2. *if $G_n[\bar{S}]$ is a disjoint union of cycles of length greater than 4 and paths, then the Chvátal rank of P is at most 2,*
3. *if $G_n[\bar{S}]$ is a forest, then the Chvátal rank of P is at most 3.*
4. *if $G_n[\bar{S}]$ has tree-width 2, then the Chvátal rank of P is at most 4.*

- Motivated by this result,

Theorem [Benchetrit, Fiorini, Huynh, Weltge, 2018]

If the tree-width of $G_n[\bar{S}]$ is t, then the Chvátal rank of P is at most $t + 2t^{t/2}$.

- Complexity results on the optimization over the Chvátal closure:

Theorem [Eisenbrand, 1999]

The separation problem over the Chvátal closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ is NP-hard.

Theorem [Cornuéjols and Li, 2016]

It is NP-hard to decide whether the Chvátal closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : \ Ax \geq b\}$ is empty, even when P contains no integer point.

Theorem [Cornuéjols, Lee, Li, 2018+] (in Chapter 2)

The separation problem over the Chvátal closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ is NP-hard, even when $P \subseteq [0,1]^n$.

Theorem [Cornuéjols, Lee, Li, 2018+] (in Chapter 2)

It is NP-hard to decide whether the Chvátal closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : \ Ax \geq b\}$ is empty, even when P contains no integer point and $P \subseteq [0,1]^n$.

- Given $S \subseteq \mathbb{Z}^n$ and a polyhedron $P \subseteq \text{conv}(S)$, *the S-Chvátal closure of P* is defined as

$$P_S := \bigcap_{c \in \mathbb{Z}^n} \left\{ x \in P : \underbrace{cx \geq \lceil \min_{y \in P} cy \rceil_{S,c}}_{\text{the } S\text{-Chvátal-Gomory cut}} \right\}.$$

where $\lceil \min_{y \in P} cy \rceil_{S,c} := \min \left\{ cz : cz \geq \min_{y \in P} cy, \; z \in S \right\} \geq \lceil \min_{y \in P} cy \rceil$.

Theorem [Dash, Günlük, Lee] (in Chapter 5)

Let $n_1, n_2, n_3, n_4 \in \mathbb{Z}_+$, and let T be a finite subset of $\mathbb{Z}^{n_1}$. Let

$$S = \left\{ (z^1, z^2, z^3, z^4) \in T \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3} \times \mathbb{Z}^{n_4} : \; \ell^2 \leq z^2, \; z^3 \leq u^3 \right\}$$

where $\ell^2 \in \mathbb{Z}^{n_2}$ and $u^3 \in \mathbb{Z}^{n_3}$. If $P \subseteq \text{conv}(S)$ is a rational polyhedron, then the S-Chvátal closure of P is a rational polyhedron.

- In particular, when $S = \{0,1\}^{n_1} \times \mathbb{Z}_+^{n_2} \times \mathbb{Z}^{n_3}$, P_S is a polyhedron.

- Split cuts are a generalization of the Chvátal-Gomory cuts.
- The split closure of a rational polyhedron is defined as the set of points satisfying all split cuts [Cook, Kannan, Schrijver, 1990].

Theorem [Caprara and Letchford, 2003]

The separation problem over the split closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ is NP-hard.

Theorem [Lee, 2018+] (in Chapter 3)

The separation problem over the split closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ is NP-hard, even when $P \subseteq [0,1]^n$.

Theorem [Lee, 2018+] (in Chapter 3)

It is NP-hard to decide whether the split closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : \ Ax \geq b\}$ is empty, even when P contains no integer point and $P \subseteq [0,1]^n$.

Part II (Chapters 6 – 9): On the $\tau = 2$ Conjecture

Based on

(1) Intersecting restrictions in clutters with A. Abdi and G. Cornuéjols, submitted.

(2) Cuboids, a class of clutters with A. Abdi, G. Cornuéjols, and N. Guričanová, submitted.

(3) Multipartite clutters with A. Abdi and G. Cornuéjols, in progress.

(4) Ideal vector spaces with A. Abdi and G. Cornuéjols, in progress.

- When is $\{x : Ax \geq b\}$ integral?
- When is a linear system $Ax \geq b$ totally dual integral (TDI)?
- $Ax \geq b$ is TDI if (D) has an integral optimal solution for every $w \in \mathbb{Z}^n$.

$$(P)\quad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & Ax \geq b \end{array} \qquad (D)\quad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & y^\top A = w^\top \\ & y \geq \mathbf{0} \end{array}$$

- If $Ax \geq b$ is TDI and b is integral, then $\{x : Ax \geq b\}$ is integral [Edmonds and Giles, 1977].
- When does the converse hold?

Question

Let M be a 0, 1 matrix such that $\{x : Mx \geq \mathbf{1},\ x \geq \mathbf{0}\}$ is integral. When is the system $Mx \geq \mathbf{1},\ x \geq \mathbf{0}$ TDI?

- To answer this question, we study combinatorial structures of M, as well as the geometry of the polyhedron $\{x : Mx \geq \mathbf{1},\ x \geq \mathbf{0}\}$.

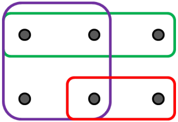
$Mx \geq \mathbf{1},\ x \geq \mathbf{0}$ where $M \in \{0,1\}^{m \times n}$.

- Let $\mathcal{C} \subseteq 2^{[n]}$ be defined as

$$\mathcal{C} := \{C \subseteq [n] : \ \chi_C \text{ is a row of } M\}.$$

- For example,

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$



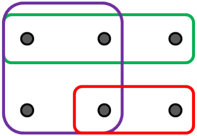
$\mathcal{C} = \{\{1,2,3\},\{5,6\},\{1,2,4,5\}\}$

- We may assume that every inequality in $Mx \geq \mathbf{1},\ x \geq \mathbf{0}$ is non-redundant.
- Sets in $\mathcal{C}$ are pairwise incomparable.

- Let E be a finite set of elements with nonnegative weights $w \in \mathbb{R}_+^E$.
- Let $\mathcal{C} \subseteq 2^E$ be a family of subsets of E, called members.
- We call $\mathcal{C}$ a clutter if the members are pairwise incomparable.
- A subset $B \subseteq E$ is a cover of $\mathcal{C}$ if

$$B \cap C \neq \emptyset \quad \forall C \in \mathcal{C}.$$

- The weight of $B \subseteq E$ is $w(B) := \sum_{e \in B} w_e$.
- The Set Covering Problem is to find a minimum weight cover of $\mathcal{C}$.



- For example, $E = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{C} = \{\{1, 2, 3\}, \{5, 6\}, \{1, 2, 4, 5\}\}$.
- $B = \{2, 5\}$ is a cover.

- Given a clutter $\mathcal{C}$, $M(\mathcal{C})$ denote the member-element incidence matrix of $\mathcal{C}$.
- We say that a clutter $\mathcal{C}$ is ideal if

$$\{x : \ M(\mathcal{C})x \geq \mathbf{1}, \ x \geq \mathbf{0}\}$$

is integral.

Examples:

1. $M(\mathcal{C})$ is totally unimodular.
2. $\mathcal{C}$ is the clutter of st-paths in a graph with distinct s, t.

$$\left\{x \in \mathbb{R}_+^E : \ x(P) \geq 1, \ \forall st\text{-path } P\right\}$$

3. $\mathcal{C}$ is the clutter of T-cuts of a graph

$$\left\{x \in \mathbb{R}_+^E : \ x(\delta(C)) \geq 1, \ \forall C \subseteq V : |C \cap T| \text{ odd}\right\}$$

- We say that a clutter $\mathcal{C}$ has the max-flow min-cut (MFMC) property if

$$M(\mathcal{C})x \geq \mathbf{1},\ x \geq \mathbf{0}$$

is total dual integral.

- $\mathcal{C}$ has the MFMC property if $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$ for any $w \in \mathbb{Z}_+^E$, where

$$\tau(\mathcal{C}, w) = \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & M(\mathcal{C})x \geq \mathbf{1} \\ & x \in \mathbb{Z}_+^E \end{array} \qquad \nu(\mathcal{C}, w) = \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & y^\top M(\mathcal{C}) \leq w^\top \\ & y \in \mathbb{Z}_+^{\mathcal{C}} \end{array}$$

- A clutter with the MFMC property is always ideal [Edmonds and Giles, 1977].

- In particular, if $\mathcal{C}$ has the MFMC property, then $\tau(\mathcal{C}) = \nu(\mathcal{C})$, where

$$\tau(\mathcal{C}) := \tau(\mathcal{C}, \mathbf{1}) = \min\left\{\mathbf{1}^\top x : \ M(\mathcal{C})x \geq \mathbf{1}, \ x \in \mathbb{Z}_+^E\right\}$$
$$\nu(\mathcal{C}) := \nu(\mathcal{C}, \mathbf{1}) = \max\left\{\mathbf{1}^\top y : \ y^\top M(\mathcal{C}) \leq \mathbf{1}^\top, \ y \in \mathbb{Z}_+^{\mathcal{C}}\right\}$$

- Notice that

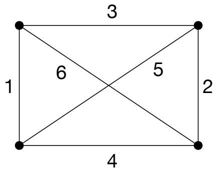
  $\tau(\mathcal{C})$ = the minimum size of a cover of $\mathcal{C}$ (covering number),

  $\nu(\mathcal{C})$ = the maximum number of disjoint members in $\mathcal{C}$ (packing number),

- We say that $\mathcal{C}$ packs if $\tau(\mathcal{C}) = \nu(\mathcal{C})$.

- However, there is an ideal clutter that does not have the MFMC property.

$$Q_6 := \{\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\}$$



$$M(Q_6) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

- Q_6 is ideal.
- $\tau(Q_6)$ = the minimum # of edges to cover all triangles = 2.
- $\nu(Q_6)$ = the maximum # of disjoint triangles = 1 $\rightarrow \tau(Q_6) > \nu(Q_6)$.

Question

When does an ideal clutter have the MFMC property?

- We define 2 minor operations with $e \in E$.

  1. Contraction $\mathcal{C}/e := \{$the minimal sets of $\{C - e : \; C \in \mathcal{C}\}\}$.
     Set w_e to a large number $\to x_e = 0$.
  2. Deletion $\mathcal{C} \setminus e := \{C \in \mathcal{C} : \; e \notin C\}$.
     Set w_e to 0 $\to x_e = 1$.

- A minor of $\mathcal{C}$ is what is obtained after a series of contractions and deletions.

Remark

1. If a clutter is ideal, then so is every minor of it.
2. If a clutter has the MFMC property, then so does every minor of it.

- In the world of ideal clutters, is there an "excluded-minor characterization" for clutters with the MFMC property?

Let $\mathcal{C}$ be a clutter.

- Recall that $\mathcal{C}$ packs if $\tau(\mathcal{C}) = \nu(\mathcal{C})$, where

  $\tau(\mathcal{C})$ = the minimum size of a cover of $\mathcal{C}$ (covering number),

  $\nu(\mathcal{C})$ = the maximum number of disjoint members in $\mathcal{C}$ (packing number).
- If $\mathcal{C}$ has the MFMC property, as the MFMC property is a minor-closed property, every minor of $\mathcal{C}$ packs.

The Replication Conjecture [Conforti and Cornuéjols, 1993]

If every minor of $\mathcal{C}$ packs, then $\mathcal{C}$ has the MFMC property.

- We say that $\mathcal{C}$ is minimally non-packing if $\mathcal{C}$ does not pack but all its proper minors pack.

The $\tau = 2$ Conjecture [Cornuéjols, Guenin, Margot, 2000]

If $\mathcal{C}$ is ideal and minimally non-packing, then $\tau(\mathcal{C}) = 2$.

- The $\tau = 2$ Conjecture $\Rightarrow$ the Replication Conjecture [Cornuéjols, Guenin, Margot, 2000].

- We say that a clutter $\mathcal{C}$ is intersecting if

$$\tau(\mathcal{C}) \geq 2 \quad \text{and} \quad \nu(\mathcal{C}) = 1.$$

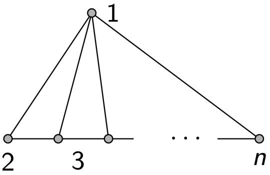
- A clutter is intersecting if any two members intersect, but there is no single common element contained in all members.
- Q_6 is intersecting, as $\tau(Q_6) = 2$ and $\nu(Q_6) = 1$.
- In fact, the $\tau = 2$ Conjecture can be equivalently stated as

The $\tau = 2$ Conjecture (version 2)

Let $\mathcal{C}$ be an ideal clutter. Then

$$\mathcal{C} \text{ has the MFMC property} \quad \Leftrightarrow \quad \mathcal{C} \text{ has no intersecting minor.}$$

- Deltas

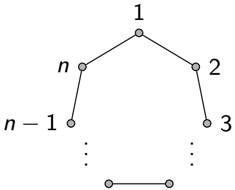


$$\begin{pmatrix} 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \\ & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$\Delta_n := \{\{1,2\},\{1,3\},\dots,\{1,n\},\{2,3,\dots,n\}\}, \quad n \geq 3$$

- Δ_n denotes the delta of dimension n.
- $\tau(\Delta_n) = 2$ and $\nu(\Delta_n) = 1$.

- The blockers of odd holes



$$\begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ 1 & & & & & 1 \end{pmatrix}$$

$$C_n^2 := \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}\,, \quad n : \text{odd}$$

- C_n^2 denotes the odd hole of dimension n.
- Every vertex cover of C_n^2 has $> \frac{n}{2}$ vertices.
- Two vertex covers of C_n^2 always intersect!
- The clutter of minimal vertex covers of C_n^2 is intersecting.

- Recall that

The $\tau = 2$ Conjecture (version 2)

Let $\mathcal{C}$ be an ideal clutter. Then

$$\mathcal{C} \text{ has the MFMC property} \quad \Leftrightarrow \quad \mathcal{C} \text{ has no intersecting minor.}$$

- Testing whether a clutter is intersecting is easy.
- However, there are $3^{|E|}$ minors.

Theorem [Abdi, Cornuéjols, Lee] in Chapter 6

Let $\mathcal{C}$ be a clutter over ground set E. One can test whether $\mathcal{C}$ contains an intersecting minor in poly$(|\mathcal{C}|, |E|)$ time.

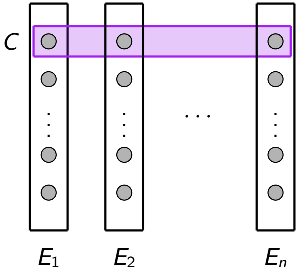
Theorem [Abdi, Cornuéjols, Lee] in Chapter 6

Let $\mathcal{C}$ be a clutter over ground set E. Then the following statements are equivalent.

(1) $\mathcal{C}$ contains an intersecting minor,

(2) There are 3 distinct members C_1, C_2, C_3 such that the minor obtained after deleting $V - (C_1 \cup C_2 \cup C_3)$ and contracting elements in covers of size 1 is intersecting.

- A *multipartite* clutter is the clutter of hyperedges in a multipartite hypergraph.



- A clutter $\mathcal{C}$ over ground set E is multipartite if E is partitioned into parts $E_1, \ldots, E_n$ so that for every $C \in \mathcal{C}$,

$$|C \cap E_i| = 1 \text{ for } i = 1, \ldots, n.$$

- $E_1, \ldots, E_n$ are covers of $\mathcal{C}$.

Question

Is there an ideal minimally non-packing multipartite clutter with large parts?

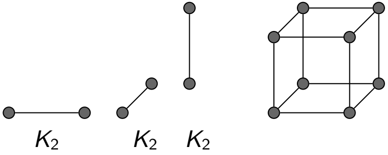
- (The $\tau = 2$ Conjecture) If a clutter $\mathcal{C}$ is ideal and minimally non-packing, then $\tau(\mathcal{C}) = 2$.
- Checking all minors is computationally expensive.
- In fact, we have shown that the $\tau = 2$ Conjecture is equivalent to the following conjecture:

Conjecture (version 3)

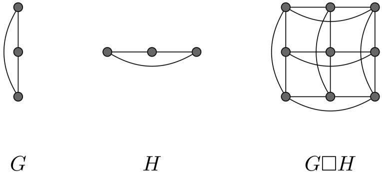
If a multipartite clutter is ideal and has no intersecting minor, then it packs.

- We have a poly-time algorithm for recognizing intersecting minors [Abdi, Cornuéjols, Lee].
- We just check if a multipartite clutter packs.
- Moreover, multipartite clutters have special structures!
- Can we find a counter-example to this conjecture?

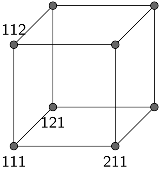
- There is another way to represent multipartite clutters as graphs.
- (The skeleton graph of) the n-dimensional hypercube is $\underbrace{K_2 \square K_2 \square \cdots \square K_2}_{n}$.



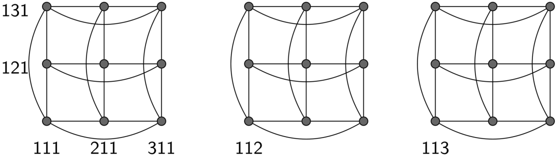
- The operation $\square$ is called the Cartesian product.
- In general, $K_{\omega_1} \square K_{\omega_2} \square \cdots \square K_{\omega_n}$ for any $\omega_1, \ldots, \omega_n \geq 1$.



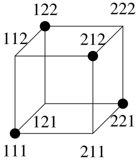
- For $n \geq 1$, $\omega_1, \ldots, \omega_n \geq 1$, let $H_{\omega_1,\ldots,\omega_n}$ denote $K_{\omega_1} \square K_{\omega_2} \square \cdots \square K_{\omega_n}$.
- $V(H_{\omega_1,\ldots,\omega_n})$ can be written as $[\omega_1] \times [\omega_2] \times \cdots \times [\omega_n]$.
- For example, $H_{\underbrace{2,\ldots,2}_{n}}$ is the n-dimensional hypercube.



- $H_{3,3,3}$ is illustrated as follows:



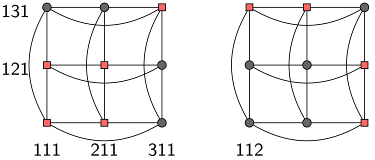
- Given $S \subseteq V(H_{\omega_1,\ldots,\omega_n}) = [\omega_1] \times [\omega_2] \times \cdots \times [\omega_n]$, one can construct a multipartite clutter associated with S, denoted $\mathrm{mult}(S)$!
- For instance, consider



$$R_{1,1} = \left\{ \begin{array}{c} 111, \\ 122, \\ 212, \\ 221 \end{array} \right\} \qquad M(\mathrm{mult}(R_{1,1})) = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array} \right]$$

$$\mathrm{mult}(R_{1,1}) = \{\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\} = Q_6.$$

- Another example is



$$S = \left\{ \begin{array}{l} 131, 231, 311, 321, \\ 112, 122, 212, 222, 332 \end{array} \right\}$$

$$M(\text{mult}(S)) = \left[\begin{array}{ccc|ccc|cc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{array} \right]$$

$$\text{mult}(S) = \{\{1, 6, 7\}, \{2, 6, 7\}, \ldots, \{3, 6, 8\}\}.$$

- In fact, every multipartite clutter can be represented as $\text{mult}(S)$ for some $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$, $\omega_1, \ldots, \omega_n \geq 1$, $n \geq 1$.

- Remember that the $\tau = 2$ Conjecture is equivalent to

The $\tau = 2$ Conjecture (version 3)

If a multipartite clutter is ideal and has no intersecting minor, then it packs.

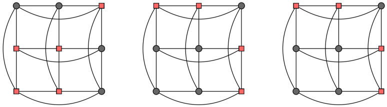
- Is there $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ such that

  (1) $\text{mult}(S)$ is ideal,

  (2) $\text{mult}(S)$ has no intersecting minor, but

  (3) $\text{mult}(S)$ does not pack?

- Given $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$, we refer to the points in S as the **feasible** points and the points in $\overline{S} := V(H_{\omega_1,\ldots,\omega_n}) \setminus S$ as the infeasible points.
- For example, in $H_{3,3,3}$, the **black** points are **feasible** and the red points are infeasible:



- The degree of S is defined as the maximum number of infeasible neighbors of an infeasible vertex.
- The degree of $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ is at most $\sum_{i=1}^{n}(\omega_i - 1)$.

Theorem [Abdi, Cornuéjols, Lee] (in Chapter 7)

Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ be of degree k. Then every minimally non-ideal minor of mult(S), if any, has at most k elements.

Corollary

Let $S \subseteq V(H_{3,3,3})$. If mult(S) is non-ideal, then it has one of Δ_3, C_5^2, $b(C_5^2)$ as a minor.

(2) Testing whether mult(S) packs

- For $u, v \in V(H_{\omega_1,\dots,\omega_n}) = [\omega_1] \times \cdots \times [\omega_n]$, the distance between u and v is equal to the number of different coordinates.
- The distance is at most n (at most n different coordinates).
- The members corresponding to u, v are disjoint if, and only if, u and v are at distance n.
- $\nu(\text{mult}(S))$ is the maximum number of points that are at pairwise distance n.

- Recall that

Theorem [Abdi, Cornuéjols, Lee] in Chapter 6

Let $\mathcal{C}$ be a clutter over ground set E. Then the following statements are equivalent.

(1) $\mathcal{C}$ contains an intersecting minor,

(2) There are 3 distinct members C_1, C_2, C_3 such that the minor obtained after deleting $V - (C_1 \cup C_2 \cup C_3)$ and contracting elements in covers of size 1 is intersecting.

- This implies

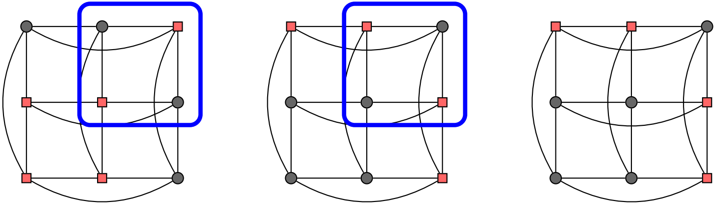
Corollary

Let $S \subseteq V(H_{\omega_1,\dots,\omega_n})$. Then the following statements are equivalent:

(1) mult(S) has no intersecting minor,

(2) there are 3 distinct points $u, v, w \in S$ such that the smallest restriction of S containing u, v, w has two points that differ in every coordinate.

- For example,



- This restriction corresponds is isomorphic to $R_{1,1}$, and $\mathrm{mult}(R_{1,1}) = Q_6$ is intersecting.

- Remember that the $\tau = 2$ Conjecture is equivalent to

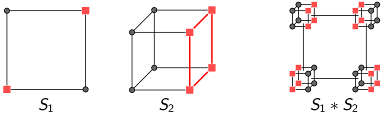
The $\tau = 2$ Conjecture (version 3)

If a multipartite clutter is ideal and has no intersecting minor, then it packs.

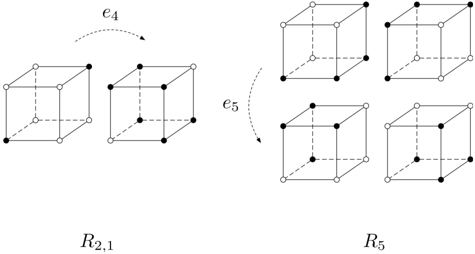
Theorem [Abdi, Cornuéjols, Lee] in Chapter 7

Let $\mathcal{C}$ be a multipartite clutter over at most 9 elements. If $\mathcal{C}$ is ideal and has no intersecting minor, then $\mathcal{C}$ packs.

- Given $S_1 \subseteq V(H_{\omega_1,\ldots,\omega_{n_1}})$ and $S_2 \subseteq V(H_{\delta_1,\ldots,\delta_{n_2}})$, the *reflective product* of S_1 and S_2 is obtained by replacing each point in S_1 with a copy of S_2 and replacing each point in $\overline{S_1}$ with a copy of $\overline{S_2}$.
- For example,



- Another example is



- Let $S_1 * S_2$ denote the reflective product of S_1 and S_2.
- Why do we care?

Theorem [Abdi, Cornuéjols, Lee] in Chapter 8

If $mult(S_1)$, $mult(\overline{S_1})$, $mult(S_2)$, $mult(\overline{S_2})$ *are ideal, then*

$$mult(S_1 * S_2), \quad mult(\overline{S_1 * S_2})$$

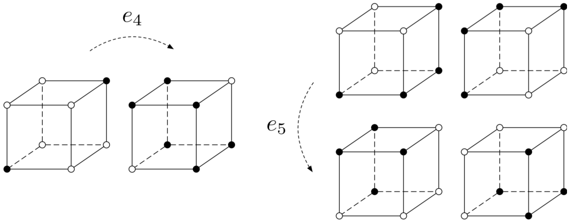
are ideal.

- One can potentially create a large class of ideal clutters using the reflective product.
- Is there a counter-example to the $\tau = 2$ Conjecture that is obtained by a reflective product of two multipartite clutters?

Theorem [Abdi, Cornuéjols, Lee] in Chapter 8

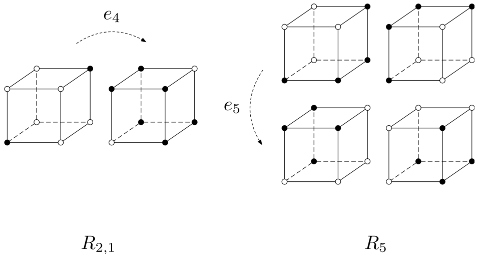
Let $S \subseteq V(H_{\omega_1,\dots,\omega_n})$. If S is the reflective product of two smaller sets and $mult(S)$ is ideal minimally non-packing, then $\omega_1 = \cdots = \omega_n = 2$ and therefore $\tau(mult(S)) = 2$.

- In fact, when $\omega_1 = \cdots = \omega_n = 2$, there are examples.



$R_{2,1}$ R_5

- When $\omega_1 = \cdots = \omega_n = 2$,



Theorem [Abdi, Cornuéjols, Guričanová, Lee] in Chapter 8

Let $S \subseteq V(H_{2,\ldots,2})$. Assume that $S = S_1 * S_2$. If mult(S) is ideal minimally non-packing, then

1. $S_1 * S_2 \cong R_{k,1}$ *for some* $k \geq 1$,
2. $n_1 = 1$ *and* $S_2, \overline{S_2}$ *are antipodally symmetric and strictly connected, or*
3. $n_2 = 1$ *and* $S_1, \overline{S_1}$ *are antipodally symmetric and strictly connected.*

- Let q be a prime power, and $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Then
$$S = \{x \in GF(q)^n : \; Ax = \mathbf{0}\}$$
for some matrix A whose entries are in $GF(q)$.
- When $q = 2$, S is called a binary space.
- As $GF(q)^n \cong [q]^n$, one can define $\text{mult}(S)$.
- (Question 1) When is $\text{mult}(S)$ ideal?
- (Question 2) When does $\text{mult}(S)$ have the max-flow min-cut property?
- Answers to these questions are provided in Chapter 9.
- For each prime power q, we have found a structural characterization and an excluded-minor characterization of when $\text{mult}(S)$ is ideal and when $\text{mult}(S)$ has the max-flow min-cut property.

Thank you!