IE 539 Convex Optimization Assignment 4

Fall 2023

Out: 13th November 2023

Due: 27th November 2023 at 11:59pm

Instructions

- Submit a PDF document with your solutions through the assignment portal on KLMS by the due date. Please ensure that your name and student ID are on the front page.
- Late assignments will be subject to a penalty. Special consideration should be applied for in this case.
- It is **required** that you typeset your solutions in LaTeX. Handwritten solutions will not be accepted.
- Spend some time ensuring your arguments are **coherent** and your solutions **clearly** communicate your ideas.

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100

1. (20 points) Let $h : \mathbb{R}^d \to \mathbb{R}$ be a closed convex function. Show that for any $x, y \in \mathbb{R}^d$ and $\eta > 0$,
$$\|\mathrm{prox}_{\eta h}(x) - \mathrm{prox}_{\eta h}(y)\|_2 \leq \|x - y\|_2.$$

2. (20 points) Let $f : \mathbb{R}^d \to \mathbb{R}$ be given by $f = g + h$ where $g : \mathbb{R}^d \to \mathbb{R}$ is a smooth convex function and $h : \mathbb{R}^d$ is closed and convex. Show that for any $\eta > 0$, $x^* \in \arg\min_{x \in \mathbb{R}^d} f(x)$ if and only if
$$x^* = (I + \eta \partial h)^{-1}(I - \eta \nabla g)(x^*).$$

3. In this question, we use Lagrangian duality to derive a solution to the following optimization problem.
$$\min_{x \in \Delta_d} \left\{ v^\top x + \sum_{i=1}^d x_i \log x_i \right\}$$
where $\Delta_d = \{x \in \mathbb{R}^d_+ : \sum_{i=1}^d x_i = 1\}$.
   (a) (10 points) The Lagrangian function is defined as
$$\mathcal{L}(x, \lambda, \mu) = v^\top x + \sum_{i=1}^d x_i \log x_i - \lambda^\top x + \mu(1 - \sum_{i=1}^d x_i).$$
   Then show that the associated Lagrangian dual function is given by
$$q(\lambda, \mu) = \mu - \sum_{i=1}^d e^{\lambda_i + \mu - v_i - 1}.$$
   (b) (5 points) Let (λ^*, μ^*) be an optimal solution to the Lagrangian dual problem. Then show that $\lambda^* = 0$ and μ^* satisfies
$$e^{\mu^* - 1} = \frac{1}{\sum_{i=1}^d e^{-v_i}}.$$
   (c) (5 points) Show that the optimal solution x^* satisfies
$$x_j^* = \frac{e^{-v_j}}{\sum_{i=1}^d e^{-v_i}} \quad \text{for } j = 1, \ldots, d.$$

4. In this question we prove the convergence of the primal-dual subgradient method for saddle point problems. Let $\phi : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be a function such that $\phi(x, y)$ for any fixed $y \in \mathbb{R}^m$ is convex in x and $\phi(x, y)$ for any fixed $x \in \mathbb{R}^d$ is concave in x. Recall that the primal-dual subgradient method proceeds as follows.
   - Choose $x_1 \in X$ and $y_1 \in Y$.
   - For $t = 1, 2, 3, \ldots, T - 1$:
     - Select $g_{x,t} \in \partial_x \phi(x_t, y_t)$, $g_{y,t} \in \partial_y \phi(x_t, y_t)$, and step size $\eta_t > 0$.
     - Compute $x_{t+1} = \mathrm{proj}_X\{x_t - \eta_t g_{x,t}\}$ and $y_{t+1} = \mathrm{proj}_Y\{y_t + \eta_t g_{y,t}\}$.

   Assume that X and Y are convex.
   (a) (5 points) Show that for any $(\bar{x}, \bar{y}) \in X \times Y$, $g_x \in \partial_x \phi(\bar{x}, \bar{y})$, and $g_y \in \partial_y \phi(\bar{x}, \bar{y})$,
$$\phi(\bar{x}, y) - \phi(x, \bar{y}) \leq -g_x^\top (x - \bar{x}) + g_y^\top (y - \bar{y}) \quad \forall (x, y) \in X \times Y.$$
   (b) (15 points) Let $\bar{x}_T$ and $\bar{y}_T$ be defined as
$$\bar{x}_T = \left(\sum_{t=1}^T \eta_t\right)^{-1} \sum_{t=1}^T \eta_t x_t, \quad \bar{y}_T = \left(\sum_{t=1}^T \eta_t\right)^{-1} \sum_{t=1}^T \eta_t y_t.$$
   Show that for any $(x, y) \in X \times Y$,
$$\phi(\bar{x}_T, y) - \phi(x, \bar{y}_T) \leq \frac{1}{2\sum_{t=1}^T \eta_t} \left(\|(x_1, y_1) - (x, y)\|_2^2 + \sum_{t=1}^T \eta_t^2 \|(g_{x,t}, g_{y,t})\|_2^2 \right).$$

5. (20 points) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a closed convex function. Using the fact that
$$x = \mathrm{prox}_f(x) + \mathrm{prox}_{f^*}(x),$$
show that for any $\lambda > 0$,
$$x = \mathrm{prox}_{\lambda f}(x) + \lambda \mathrm{prox}_{(1/\lambda) f^*}(x/\lambda).$$