Introduction to Modern Combinatorial Optimization

The Center for Algorithm and Optimization (CALOP), POSTECH

Winter 2025

Instructor:

Dabeen Lee, dabeenl@kaist.ac.kr, Department of Industrial and Systems Engineering, KAIST.

Lecture schedule:

12 lectures in the week of January 13 - 17, 2025

Classroom:

- January 13-15 & 17: School of Environmental Science and Engineering Building, #101
- January 16: LG Cooperative Electronics Engineering Building, #105

https://calop.postech.ac.kr/event/2024-25-calop-winter-lecture/.

Course webpage:

https://dabeenl.github.io/Comb_Opt (*Lecture notes will be uploaded to the webpage.*)

Course description One of the early developments in combinatorial optimization dates back to 1784 when G. Monge studied the assignment problem in the context of transporting molecules. Later, in the early 20th century, pioneering works on bipartite matching and linear programming came to light. Since then, innovations in combinatorial optimization have revolutionized all areas of operations research and computer science. To date, algorithms and techniques inspired by combinatorial optimization have played a central role in scientific fields including machine learning, artificial intelligence, market design economics, protein structure prediction, and drug design. Motivated by this, this course aims to equip students with an in-depth understanding of comprehensive aspects of combinatorial optimization. We will cover a wide range of topics, from classic results to modern approaches, about fundamental theory and practical algorithms.

Key topics Comprehensive, while advanced, topics are covered in depth from a few important problems in combinatorial optimization. This course offers an overview of modern combinatorial optimization, discussing algorithms, mathematical programming techniques, and a bit of machine learning theory.

- Algorithms: maximum bipartite & non-bipartite matching, the Gale-Shapley algorithm for stable matching, the VCG mechanism for matching markets, online bipartite matching, algorithms for submodular function maximization
- Mathematical programming: duality, mininax theorems, bipartite & non-bipartite matching polytopes, primal-dual methods, polymatroid optimization, chance-constrained programming

- Machine learning: first-order methods, online submodular maximization, Bayesian optimization, neural combinatorial optimization

Textbook There is no required textbook for this course, but students may find the following list of materials useful to follow the topics covered in this course:

- On the History of Combinatorial Optimization (Till 1960), by Alexander Schrijver

  `https://www.sciencedirect.com/science/article/abs/pii/S0927050705120015`

- Introduction to Graph Theory, by Douglas B. West,

  `https://dwest.web.illinois.edu/igt/`

- Combinatorial Optimization: Polyhedra and Efficiency, by Alexander Schrijver,

  `https://link.springer.com/book/9783540443896`

- Integer Programming, by Michele Conforti, Gérald Cornuéjols, and Giacomo Zambelli,

  `https://link.springer.com/book/10.1007/978-3-319-11008-0`

- Submodular Function Maximization, by Andreas Krause and Daniel Golovin,

  `https://viterbi-web.usc.edu/~shanghua/teaching/Fall2021-670/krause12survey.pdf`

Prerequisites There are no formal prerequisites but you should be comfortable making mathematical arguments and writing proofs. You should also be comfortable with the background knowledge from previous courses. Such topics include

- linear programming and graph theory,
- basic linear algebra (vectors, matrices, inner products),
- basic complexity theory (time complexity, polynomial algorithms, P vs NP).

Course outline What follows is a tentative, but maybe too ambitious, outline of this course.

1. Bipartite matching
   - Hall's marriage theorem
   - Augmenting path algorithm for maximum bipartite matching
   - König's minimax theorem on maximum bipartite matching and minimum vertex cover
   - Deriving König's theorem from Menger's theorem on shortest paths and minimum cuts
   - Bipartite matching polytope and duality
   - Hungarian method for maximum weight bipartite matching from a primal-dual perspective
   - Stable matching: the Gale-Shapley algorithm and the polyhedral description

- Matching markets: the Vickrey–Clarke–Groves (VCG) mechanism
- Online bipartite matching

2. Non-bipartite matching
    - Edmonds' blossom algorithm for maximum matching
    - Matching polytope and maximum weight matching
3. Submodular function minimization
    - Polymatroid optimization
    - Lovász extension for submodular functions
    - Applications to chance-constrained programming
4. Submodular function maximization
    - Greedy algorithm for monotone submodular function maximization
    - Submodular welfare maximization
    - Submodular function maximization subject to a matroid constraint
    - First-order methods: gradient ascent and the Frank-Wolfe algorithm
    - Online submodular maximization
5. Combinatorial optimization aided by machine learning
    - Bayesian optimization and metaheuristic algorithms
    - Neural combinatorial optimization