1 Outline

In this lecture, we cover

- the two-phase simplex algorithm,
- recognizing infeasible and unbounded linear programs.
- simplex method technical details.

2 Two-Phase Simplex Algorithm

In the last lecture, we applied the simplex method to solve the following linear program with two variables.

$$\begin{aligned} \max \quad & z = 5x + 4y \\ \text{s.t.} \quad & 2x + 3y \leq 150, \\ & 2x + y \leq 70, \\ & x, y \geq 0. \end{aligned}$$

This linear program can be converted into standard form, given by

$$\begin{aligned} \max_{x,y} \quad & z = 5x + 4y \\ \text{s.t.} \quad & 2x + 3y + s_1 = 150, \\ & 2x + y + s_2 = 70, \\ & x, y, s_1, s_2 \geq 0. \end{aligned}$$

Recall that the slack variables s_1 and s_1 naturally give rise to the initial dictionary as follows.

$$\begin{array}{rclccc} z & = & & +5x & +4y, \\ s_1 & = & 150 & -2x & -3y, \\ s_2 & = & 70 & -2x & -y. \end{array}$$

Then we obtain the initial solution $(x, y, s_1, s_2) = (0, 0, 150, 70)$. We say that this is a **feasible dictionary**.

Note that using the slack variables for the initial dictionary was feasible because the right-hand side values, 150 and 70, are all nonnegative. What about the following linear program?

$$\begin{aligned} \max \quad & z = x + 2y \\ \text{s.t.} \quad & 2x + 3y \leq 150, \\ & -x + y \leq -25, \\ & x, y \geq 0. \end{aligned}$$

As before, we use slack variables s_1 and s_2 to transform the inequality constraints into equalities as follows.

$$\begin{aligned} \max \quad & z = x + 2y \\ \text{s.t.} \quad & 2x + 3y + s_1 = 150, \\ & -x + y + s_2 = -25, \\ & x, y \geq 0, \end{aligned}$$

and this gives rise to the following dictionary.

$$\begin{array}{rcrrr} z & = & & +x & +2y, \\ s_1 & = & 150 & -2x & -3y, \\ s_2 & = & -25 & +x & -y. \end{array}$$

Here, if we set $x = y = 0$, then (s_1, s_2) would be $(150, -25)$, violating the nonnegativity constraint on s_2. We say that this is an **infeasible dictionary**. We cannot proceed the simplex algorithm with an infeasible dictionary.

Motivated by this, the **two-phase** simplex algorithm proceeds in the phase of finding a feasible dictionary and the solution phase. In the first phase, we check the feasibility of the problem. If the given linear program is feasible, then the first phase ends with a feasible dictionary. If not, we conclude that the linear program is infeasible. The solution phase is the simplex algorithm outlined in the last lecture.

2.1 Phase I: Finding a Feasible Dictionary

For the first phase, we consider another linear program to check the feasibility of the original linear program. For our example, we consider

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & 2x + 3y - t \leq 150, \\ & -x + y - t \leq -25, \\ & x, y, t \geq 0. \end{aligned}$$

Theorem 5.1. *The linear program is feasible and its optimal value is equal to 0 if and only if the original linear program is feasible.*

Proof. We may take a sufficiently large number for t to make the constraints always satisfied. Moreover, the optimal value is 0 if and only if there is a solution (x, y, t) with $t = 0$ that satisfies the constraints. $\square$

The standard form is given by

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & 2x + 3y - t + s_1 = 150, \\ & -x + y - t + s_2 = -25, \\ & x, y \geq 0, \end{aligned}$$

Then the corresponding initial dictionary is given by

$$\begin{array}{rcrrrr} z & = & & & & -t, \\ s_1 & = & 150 & -2x & -3y & +t, \\ s_2 & = & -25 & +x & -y & +t. \end{array}$$

Here, we have $-t$ in the objective row because minimizing t is equivalent to maximizing $-t$. Again, this dictionary is infeasible, but we may obtain a feasible dictionary with the new variable t. Variable t becomes basic, while a basic variable with the most negative value becomes non-basic. For the current dictionary, s_2 has the only variable with a negative value. Before moving t to the left-hand side, we apply row operations to eliminate t from the other rows.

$$\begin{array}{rcllll} z+s_2 &=& -25 & +x & -y & \\ s_1-s_2 &=& 175 & -3x & -2y & \\ s_2 &=& -25 & +x & -y & +t. \end{array}$$

Then we move t to the left-hand side and s_2 to the right-hand side.

$$\begin{array}{rcllll} z &=& -25 & +x & -y & -s_2 \\ s_1 &=& 175 & -3x & -2y & +s_2 \\ t &=& 25 & -x & +y & +s_2. \end{array}$$

Next, we note that x has a positive coefficient in the objective row. Then we may increase x up to

$$\min\left\{\frac{175}{3},\ 25\right\} = 25,$$

in which case t becomes 0 and becomes non-basic. Equivalently, we move x to the left-hand side and t to the right-hand side. Before this, we apply row operations as follows.

$$\begin{array}{rcllll} z+t &=& & & & \\ s_1-3t &=& 100 & & -5y & -2s_2 \\ t &=& 25 & -x & +y & +s_2. \end{array}$$

Then we deduce

$$\begin{array}{rcllll} z &=& & -t & & \\ s_1 &=& 100 & +3t & -5y & -2s_2 \\ x &=& 25 & -t & +y & +s_2. \end{array}$$

As all the coefficients in the objective row are nonpositive, this dictionary is optimal. Moreover, the optimal value is 0. Therefore, the original linear program is feasible, and the dictionary with basic variables s_1 and s provides a feasible dictionary. The feasible dictionary is given by

$$\begin{array}{rcll} s_1 &=& 100-5y & -2s_2 \\ x &=& 25+y & +s_2. \end{array}$$

Note that we are missing the objective row. In fact, the objective z is given by $z = x + 2y$, and we can combine this with the dictionary. As x is a basic variable, we replace x with non-basic variables.

$$z = (25 + y + s_2) + 2y = 25 + s_2 + 3y.$$

Then we obtain

$$\begin{array}{rclll} z &=& 25 & +s_2 & +3y, \\ s_1 &=& 100 & -2s_2 & -5y, \\ x &=& 25 & +s_2 & +y. \end{array}$$

The first feasible dictionary gives rise to the initial solution

$$(x, y) = (25, 0) \quad (s_1, s_2) = (100, 0).$$

2.2 Phase II: Running the Simplex Method with a Feasible Dictionary

There are variables with positive coefficients in the objective row from the first feasible dictionary. In particular, y has a strictly positive coefficient. Let us make y basic. Then we may increase y up to 20, which would make s_1 non-basic. By applying the required row operations, we obtain

$$\begin{array}{rcllr} z + 0.6s_1 & = & 85 & -0.2s_2 & \\ s_1 & = & 100 & -2s_2 & -5y, \\ x + 0.2s_1 & = & 45 & +0.6s_2 & \end{array}$$

Moving s_1 to the right-hand side and y to the left-hand side, we obtain

$$\begin{array}{rclll} z & = & 85 & -0.2s_2 & -0.6s_1 \\ y & = & 20 & -0.4s_2 & -0.2s_1, \\ x & = & 45 & +0.6s_2 & -0.2s_1 \end{array}$$

Here, all objective coefficients are negative, and therefore, the currrent solution

$$(x, y) = (45, 20), \quad (s_1, s_2) = (0, 0)$$

is optimal.

2.3 Geometry

Note that the initial infeasible dictionary gave solution

$$(x, y) = (0, 0), \quad (s_1, s_2) = (150, -25).$$

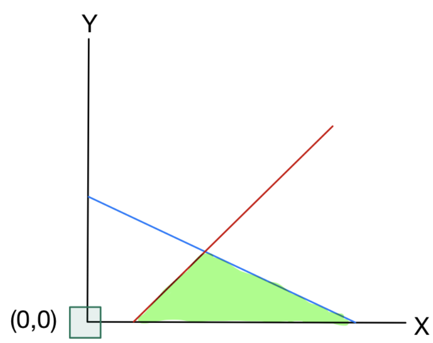

Figure 5.1: Solution from the initial infeasible dictionary

It can be checked from Figure 5.1 that $(x, y) = (0, 0)$ is an infeasible solution to the original linear program. Next, after Phase I, we obtained

$$(x, y) = (25, 0) \quad (s_1, s_2) = (100, 0)$$

from the first feasible dictionary. The first feasible solution is illustrated in the first figure in Figure 5.2. After this solution, we reached the optimality with solution

$$(x, y) = (45, 20), \quad (s_1, s_2) = (0, 0).$$

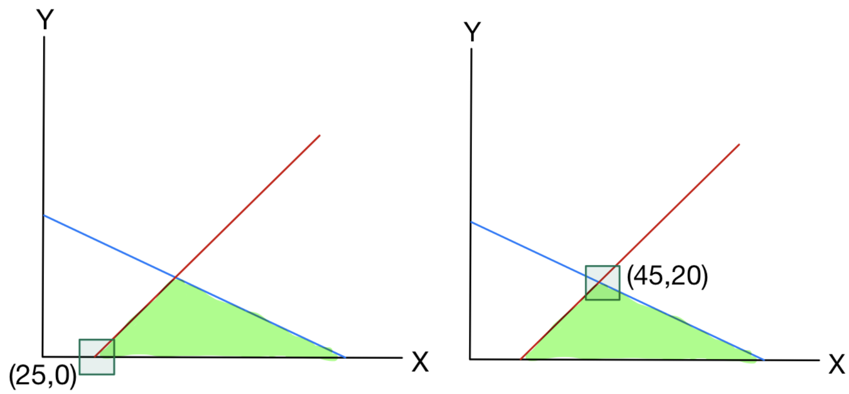

Figure 5.2: Solution from the first feasible dictionary and the optimal solution

3 Simplex Method for Infeasible and Unbounded Linear Programs

3.1 Infeasible Case

Next, we determine whether the following linear linear program is feasible or not.

$$\begin{aligned} \max \quad & z = x + 2y \\ \text{s.t.} \quad & 2x + 3y \leq 150, \\ & -x + y \leq -90, \\ & x, y \geq 0. \end{aligned}$$

As explained in the last section, we can test the feasibility by running Phase I. To do so, we consider

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & 2x + 3y - t \leq 150, \\ & -x + y - t \leq -90, \\ & x, y \geq 0, \end{aligned}$$

which gives rise to the initial dictionary.

$$\begin{array}{rclllll} z &=& & & & -t, \\ s_1 &=& 150 & -2x & -3y & +t, \\ s_2 &=& -90 & +x & -y & +t. \end{array}$$

Moving t to the left-hand side and s_2 to the right-hand side, we obtain

$$\begin{array}{rclllll} z &=& -90 & +x & -y & -s_2 \\ s_1 &=& 240 & -3x & -2y & +s_2 \\ t &=& 90 & -x & +y & +s_2. \end{array}$$

Next x becomes basic while s_1 becomes non-basic. Applying row operations,

$$\begin{array}{rclllll} z + (1/3)s_1 &=& -10 & & -(5/3)y & -(2/3)s_2 \\ (1/3)s_1 &=& 80 & -x & -(2/3)y & +(1/3)s_2 \\ t - (1/3)s_1 &=& 10 & & +(5/3)y & +(2/3)s_2. \end{array}$$

Then we obtain

$$\begin{array}{rcllll} z &=& -10 & -(1/3)s_1 & -(5/3)y & -(2/3)s_2 \\ x &=& 80 & -(1/3)s_1 & -(2/3)y & +(1/3)s_2 \\ t &=& 10 & +(1/3)s_1 & +(5/3)y & +(2/3)s_2. \end{array}$$

Now all the objective coefficients are non-positive. However, the value of t is 10 which is strictly positive. Hence, the linear program is infeasible!

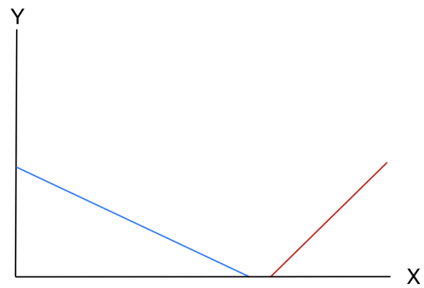

Figure 5.3: Infeasible case

3.2 Unbounded Case

Now we consider

$$\begin{aligned} \max \quad & z = x + y \\ \text{s.t.} \quad & -2x + y \le 100, \\ & x - 2y \le 100, \\ & x, y \ge 0. \end{aligned}$$

Its standard form is given by

$$\begin{aligned} \max \quad & z = x + y \\ \text{s.t.} \quad & -2x + y + s_1 = 100, \\ & x - 2y + s_2 = 100, \\ & x, y \ge 0. \end{aligned}$$

Then the initial dictionary is given by

$$\begin{array}{rclll} z &=& & +x & +y \\ s_1 &=& 100 & +2x & -y \\ s_2 &=& 100 & -x & +2y \end{array}$$

Next we make x basic and s_2 non-basic. Applying row operations,

$$\begin{array}{rclll} z + s_2 &=& 100 & & +3y \\ s_1 + 2s_2 &=& 300 & & +3y \\ s_2 &=& 100 & -x & +2y \end{array}$$

Moving s_2 to the right-hand side and x to the left-hand side, we deduce

$$\begin{array}{rclll} z &=& 100 & -s_2 & +3y \\ s_1 &=& 300 & -2s_2 & +3y \\ x &=& 100 & -s_2 & +2y \end{array}$$

Now, note that y has a positive objective coefficient. In fact, we may increase y by as much as we want without violating the nonnegativity constraints on the current basic variables! This implies that the linear program is unbounded.

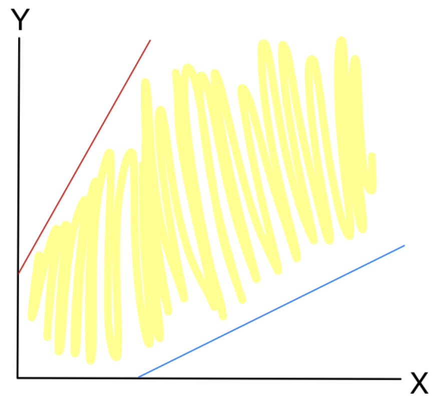

Figure 5.4: Unbounded case

4 Simplex Method Phase I Technical Details

Recall that the two-phase simplex method starts with the step of finding a feasible dictionary. This step is essentially checking the feasibility of the linear program.

4.1 Inequality Constraints

Assume that the constraints are given by

$$Ax \leq b, \quad x \geq 0.$$

Then adding slack variables s, we obtain an equivalent system

$$\begin{aligned} Ax + s = &\begin{bmatrix} a_1^\top x + s_1 \\ \vdots \\ a_m^\top x + s_m \end{bmatrix} = b \\ x \geq 0&, s \geq 0. \end{aligned}$$

This gives rise to the initial dictionary

$$\underbrace{\begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}}_{s} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{b} + \underbrace{\begin{bmatrix} -a_1^\top x \\ \vdots \\ -a_m^\top x \end{bmatrix}}_{-Ax}.$$

If the components of b are all nonnegative, then it is a feasible dictionary. If not,

$$b_{\min} = \min_{i \in [m]} b_i$$

is negative. In this case, we consider

$$Ax - t\mathbf{1} \leq b, \quad x \geq 0, t \geq 0$$

where $\mathbf{1}$ the vector of all ones. In conrast to the original system, this new system with variable t is always feasible. In fact,

$$(x,t) = (0, -b_{\min})$$

is a feasible solution. Moreover, consider

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & Ax - t\mathbf{1} \leq b \\ & x \geq 0, t \geq 0 \end{aligned}$$

which we refer to as the **Phase I LP**. We know that system $Ax \leq b, x \geq 0$ is feasible if and only if the Phase I LP has optimal value 0. That is because the optimal value being 0 means that there exists a solution $(x,t) = (\bar{x}, 0)$ that satisfies $A\bar{x} - 0\mathbf{1} = A\bar{x} \leq b$ and $\bar{x} \geq 0$.

To solve the Phase I LP, we obtain its standard form, given by

$$\begin{aligned} & Ax - t\mathbf{1} + s = \begin{bmatrix} a_1^\top x - t + s_1 \\ \vdots \\ a_m^\top x - t + s_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \\ & x \geq 0, t \geq 0, s \geq 0. \end{aligned}$$

Let us get the initial dictionary of this. For simpler presentation, let us assume that b_1 is the smallest among the components of b. Subtracting the first row from the other rows, we obtain

$$\begin{bmatrix} a_1^\top x - t + s_1 \\ (a_2 - a_1)^\top x + s_2 - s_1 \\ \vdots \\ (a_m - a_1)^\top x + s_m - s_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ \vdots \\ b_m - b_1 \end{bmatrix}.$$

Moving the variables accordingly, we get

$$\begin{bmatrix} t \\ s_2 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} -b_1 \\ b_2 - b_1 \\ \vdots \\ b_m - b_1 \end{bmatrix} + \begin{bmatrix} a_1^\top x + s_1 \\ -(a_2 - a_1)^\top x + s_1 \\ \vdots \\ -(a_m - a_1)^\top x + s_1 \end{bmatrix}.$$

Here, this dictionary is feasible because

$$b_i - b_1 = b_i - \min\{b_j : \; j \in [m]\} \geq 0$$

for all $i \in [m]$. Then we may proceed the simplex algorithm to solve the Phase I LP.

4.2 Equality Constraints

Now assume that the constraints of a linear program are given by

$$Ax = b, \quad x \geq 0.$$

In this case, what would be the right form for the Phase I LP? We may apply the same idea! We consider

$$\begin{aligned} \min \quad & t + \sum_{i=1}^m s_i \\ \text{s.t.} \quad & Ax - t\mathbf{1} + s = b \\ & x \geq 0, s \geq 0, t \geq 0. \end{aligned}$$

where $\mathbf{1}$ is the veector of all ones. Here s is the vector of m variables and t is a single variable. This is the **Phase I LP** for the equality constrained case.

If all components of b are nonnegative,

$$(x,t,s) = (0,0,b)$$

is a feasible solution! Moreover,

$$\begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} + \begin{bmatrix} -a_1^\top x + t \\ \vdots \\ -a_m^\top x + t \end{bmatrix}$$

is a feasible dictionary.

Next, consider the case when b has some negative component. WIthout loss of generality, we may assume that b_1 has the smallest value among the components of b. In this case, we use t instead of s_1 for a basic variable. Then we obtain from $Ax - t\mathbf{1} + s = b$ that

$$\begin{bmatrix} t \\ s_2 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} -b_1 \\ b_2 - b_1 \\ \vdots \\ b_m - b_1 \end{bmatrix} + \begin{bmatrix} a_1^\top x + s_1 \\ -(a_2 - a_1)^\top x + s_1 \\ \vdots \\ -(a_m - a_1)^\top x + s_1 \end{bmatrix}.$$

This is a feasible dictionary.

Theorem 5.2. *The system $Ax = b$, $x \geq 0$ is feasible if and only if the Phase I LP has optimal value 0.*

Proof. The system $Ax = b$, $x \geq 0$ is feasible if and only if there exists (x,t,s) with $t = 0$ and $s = 0$ satisfies $Ax - t\mathbf{1} + s = b$ with $x \geq 0$, in which case (x,t,s) has objective value 0. □

5 Phase II Technical Details

In this section, we review the second phase of the simplex algorithm for general linear programs. Consider a linear program in standard form as follows.

$$\begin{aligned} \max \quad & z = c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Assume that we are given a feasible dictionary, which is guaranteed after the first phase of the simplex algorithm. Suppose that the vector of variables x has **basic variables** x_B and **non-basic variables** x_N, i.e., we may write x as

$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}.$$

Moreover, we decompose the objective coefficient vector c and the constraint matrix A with respect to the basic and non-basic variables. Basically,

$$c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}, \quad A = \begin{bmatrix} B & N \end{bmatrix}.$$

Assumption 1. To set x_B as basic variables, the corresponding constraint matrix B should be nonsingular, i.e., B^{-1} exists.

Then the linear program in standard form can be rewritten as

$$\begin{aligned} \max \quad & z = c_B^\top x_B + c_N^\top x_N \\ \text{s.t.} \quad & Bx_B + Nx_N = b \\ & x_B \geq 0, x_N \geq 0. \end{aligned}$$

To obtain the feasible dictionary with respect to the basic variables x_B, we need to apply the required row operations. In fact, applying the required row operations is equivalent to multiplying both sides by B^{-1}. It follows from $Bx_B + Nx_N = b$ that

$$x_B = B^{-1}b - B^{-1}Nx_N.$$

Assumption 2. For the basic variables x_B to give a feasible dictionary, all components of $B^{-1}b$ are nonnegative.

Next we need to eliminate the basic variables x_B from the objective row. To do so, we use

$$c_B^\top x_B = c_B^\top B^{-1}b - c_B^\top B^{-1}Nx_N.$$

We plug in this to the objective row to eliminate x_B. Note that

$$\begin{aligned} z &= c_B^\top x_B + c_N^\top x_N \\ &= c_B^\top B^{-1}b - c_B^\top B^{-1}Nx_N + c_N^\top x_N \\ &= c_B^\top B^{-1}b + (c_N - N^\top (B^{-1})^\top c_B)^\top x_N. \end{aligned}$$

Consequently, the dictionary is given by

$$\begin{array}{rll} z = & c_B^\top B^{-1}b & +(c_N - N^\top (B^{-1})^\top c_B)^\top x_N \\ x_B = & B^{-1}b & \qquad\qquad\qquad\qquad -B^{-1}Nx_N \end{array}$$

- The corresponding solution is given by

$$(x_B, x_N) = (B^{-1}b, 0).$$

- The corresponding objective value is

$$z = c_B^\top B^{-1}b.$$

- We call the matrix B or the columns corresponding to the basic variables x_B **basis**.
- The objective coefficients $c_N - N^\top (B^{-1})^\top c_B$ are called the **reduced costs**.
- For maximization, the current dictionary is optimal if the reduced costs are all non-positive.