Mathematical and Numerical Optimization Assignment 2

Spring 2026

Out: 18th May 2026

Due: 25th May 2026 at 11:59pm

Instructions

- Submit a PDF document with your solutions through the assignment portal on KLMS by the due date. Please ensure that your name and student ID are on the front page.
- Late assignments will **not** be accepted except in extenuating circumstances. Special consideration should be applied for in this case.
- It is **required** that you typeset your solutions in LaTeX. Handwritten solutions will not be accepted.
- Spend some time ensuring your arguments are **coherent** and your solutions **clearly** communicate your ideas.

Question:	1	2	3	4	Total
Points:	40	20	20	20	100

1. In this question we prove the convergence of the primal-dual subgradient method for saddle point problems. Let $\phi : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be a function such that $\phi(x, y)$ for any fixed $y \in \mathbb{R}^m$ is convex in x and $\phi(x, y)$ for any fixed $x \in \mathbb{R}^d$ is concave in x. Recall that the primal-dual subgradient method proceeds as follows.

   - Choose $x_1 \in X$ and $y_1 \in Y$.
   - For $t = 1, 2, 3, \ldots, T - 1$:
     - Select $g_{x,t} \in \partial_x \phi(x_t, y_t)$, $g_{y,t} \in \partial_y \phi(x_t, y_t)$, and step size $\eta_t > 0$.
     - Compute $x_{t+1} = \mathrm{proj}_X\{x_t - \eta_t g_{x,t}\}$ and $y_{t+1} = \mathrm{proj}_Y\{y_t + \eta_t g_{y,t}\}$.

   Assume that X and Y are convex.

   (a) (10 points) Show that for any $(\bar{x}, \bar{y}) \in X \times Y$, $g_x \in \partial_x \phi(\bar{x}, \bar{y})$, and $g_y \in \partial_y \phi(\bar{x}, \bar{y})$,

   $$\phi(\bar{x}, y) - \phi(x, \bar{y}) \leq -g_x^\top (x - \bar{x}) + g_y^\top (y - \bar{y}) \quad \forall (x, y) \in X \times Y.$$

   (b) (30 points) Let $\bar{x}_T$ and $\bar{y}_T$ be defined as

   $$\bar{x}_T = \left(\sum_{t=1}^T \eta_t\right)^{-1} \sum_{t=1}^T \eta_t x_t, \quad \bar{y}_T = \left(\sum_{t=1}^T \eta_t\right)^{-1} \sum_{t=1}^T \eta_t y_t.$$

   Show that for any $(x, y) \in X \times Y$,

   $$\phi(\bar{x}_T, y) - \phi(x, \bar{y}_T) \leq \frac{1}{2\sum_{t=1}^T \eta_t} \left(\|(x_1, y_1) - (x, y)\|_2^2 + \sum_{t=1}^T \eta_t^2 \|(g_{x,t}, g_{y,t})\|_2^2 \right).$$

2. (20 points) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a closed convex function. Using the fact that

   $$x = \mathrm{prox}_f(x) + \mathrm{prox}_{f^*}(x),$$

   show that for any $\lambda > 0$,

   $$x = \mathrm{prox}_{\lambda f}(x) + \lambda \mathrm{prox}_{(1/\lambda) f^*}(x/\lambda).$$

3. (20 points) Let $h : \mathbb{R}^d \to \mathbb{R}$ be a closed convex function. Show that for any $x, y \in \mathbb{R}^d$ and $\eta > 0$,

   $$\|\mathrm{prox}_{\eta h}(x) - \mathrm{prox}_{\eta h}(y)\|_2 \leq \|x - y\|_2.$$

4. (20 points) Let $f : \mathbb{R}^d \to \mathbb{R}$ be given by $f = g + h$ where $g : \mathbb{R}^d \to \mathbb{R}$ is a smooth convex function and $h : \mathbb{R}^d$ is closed and convex. Show that for any $\eta > 0$, $x^* \in \arg\min_{x \in \mathbb{R}^d} f(x)$ if and only if

   $$x^* = (I + \eta \partial h)^{-1} (I - \eta \nabla g)(x^*).$$