3341.454 Mathematical and Numerical Optimization Assignment 1

Spring 2026

Out: 30th March 2026

Due: 8th April 2026 at 11:59pm

Instructions

- Submit a PDF document with your solutions through the assignment portal on KLMS by the due date. Please ensure that your name and student ID are on the front page.
- Late assignments will **not** be accepted except in extenuating circumstances. Special consideration should be applied for in this case.
- It is **required** that you typeset your solutions in LaTeX. Handwritten solutions will not be accepted.
- Spend some time ensuring your arguments are **coherent** and your solutions **clearly** communicate your ideas.

Question:	1	2	3	4	5	Total
Points:	15	20	20	30	15	100

1. (15 points) Represent the projection S of the polyhedral set

$$Y = \left\{x \in \mathbb{R}^3 : -1 \leq x_1 + x_2 \leq 1, -1 \leq x_2 - x_3 \leq 1, -1 \leq x_1 + x_3 \leq 1, x_1 + x_2 - x_3 \leq 1\right\}$$

onto the x_1, x_2-plane by a system of linear inequalities in the variables x_1 and x_2.

2. Consider the problem

$$\begin{aligned} \min_{x \in \mathbb{R}^d} \quad & \frac{c^\top x + d}{f^\top x + g} \\ \text{s.t.} \quad & Ax \leq b \\ & f^\top x + g > 0 \end{aligned}$$

(a) (10 points) Show that the problem is equivalent to the following linear program:

$$\begin{aligned} \min_{x \in \mathbb{R}^d, t \in \mathbb{R}} \quad & c^\top x + dt \\ \text{s.t.} \quad & Ax \leq bt \\ & f^\top x + gt = 1 \\ & t > 0. \end{aligned}$$

(b) (10 points) Based on this, provide a procedure that allows us to compute the optimal value of the original problem by solving a finite number of linear programs.

3. (20 points) Consider a polyhedron P given in inequality form, i.e.,

$$P = \{x \in \mathbb{R}^n : a_i^\top x \leq b, i = 1, \ldots, m\}.$$

An Euclidean ball of radius r centered at y is given by

$$B(y, r) = \{x \in \mathbb{R}^n : \|x - y\|_2 \leq r\}.$$

We are interested in finding the largest ball contained in P. Formulate this problem as a linear program.

4. Verify convexity/concavity of the following functions. You may use the first-order and second-order characterizations of convex functions, while there exists a direct proof based on the definition of convex functions.

(a) (5 points) The *negative entropy function* is convex on $\mathbb{R}^d_{++}$:

$$f(x) := \sum_{i \in [d]} x_i \log(x_i).$$

(b) (5 points) The *log-sum-exp function* is convex:

$$f(x) = \log\left(\sum_{i \in [d]} \exp(x_i)\right).$$

[Hint: An elementary proof exists by showing $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$. You may use (without proof) the inequality $\sum_{i \in [d]} |u_i|^\lambda |v_i|^{1-\lambda} \leq \left(\sum_{i \in [d]} |u_i|\right)^\lambda \left(\sum_{i \in [d]} |v_i|\right)^{1-\lambda}$.]

(c) (5 points) The geometric mean is concave on $\mathbb{R}^d_{++}$:

$$f(x) = \left(\prod_{i \in [d]} x_i\right)^{1/d}.$$

(d) (5 points) The log-determinant is concave on $\mathbb{S}^d_{++}$, the set of $d \times d$ positive "definite" matrices:

$$f(X) = \log\det(X).$$

Note: the determinant of a matrix $X \in \mathbb{S}^d_{++}$ is simply the product of its eigenvalues (which are all positive by assumption). You will need the following properties of matrices/determinants which you can use without proof:

- $\det(AB) = \det(BA) = \det(A)\det(B)$ for $A, B \in \mathbb{S}^d_{++}$.
- If $A \in \mathbb{S}^d_{++}$, then we can write $A = PDP^\top$ where $P^\top P = PP^\top = I$ and D is diagonal with strictly positive entries. Let D^r be the diagonal matrix with all diagonal entries of D raised to the power $r \in \mathbb{R}$. We can define powers of A via $A^r = PD^rP^\top$, which have the same properties as the usual powers: $A^rA^s = A^sA^r = A^{r+s}$, $A^0 = I$. Furthermore, $\det(A^r) = \det(A)^r$.
- $\alpha A + \beta B = A^{1/2}(\alpha I + \beta A^{-1/2}BA^{-1/2})A^{1/2}$ for any $A, B \in \mathbb{S}^d_{++}$ and $\alpha, \beta \geq 0$.

(e) (5 points) The *conjugate* of a function $f : \mathbb{R}^d \to \mathbb{R}$:

$$f^*(x) = \sup_{y \in \mathbb{R}^d} \{\langle y, x\rangle - f(y)\}$$

(f) (5 points) The sum of k largest components of $x \in \mathbb{R}^d$:

$$f(x) = x_{\sigma(1)} + \cdots + x_{\sigma(k)}$$

where $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(d)}$ are the rearrangement of $x_1, \ldots, x_d$ in nonincreasing order.

5. Let $K = \{(x, y, z) \in \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R} : y\exp(x/y) \leq z\}$.

(a) (5 points) Let $K = \{(x, y, z) \in \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R} : y\exp(x/y) \leq z\}$. Show that K is a convex cone.

(b) (2 points) The dual cone of K is given by

$$K^* = \left\{(u, v, w) \in \mathbb{R}^3 : ux + vy + wz \geq 0 \ \forall (x, y, z) \in K\right\}.$$

Show that for any $(u, v, w) \in K^*$ with $w = 0$, the point satisfies $u = 0$ and $v \geq 0$. Moreover, argue that $\{(0, v, 0) : v \geq 0\} \subset K^*$.

(c) (2 points) Show that for any $(u, v, w) \in K^*$ with $w > 0$ and $u = 0$, the point satisfies $v \geq 0$. Moreover, argue that $\{(0, v, 0) : v \geq 0\} \subset K^*$. Moreover, argue that $\{(0, v, w) : v \geq 0, w > 0\} \subset K^*$.

(d) (2 points) Show that for any $(u, v, w) \in K^*$ with $w > 0$ and $u \neq 0$, the point satisfies $u < 0$.

(e) (4 points) Finish the proof that the dual cone of K from part (b) is given by

$$K^* = \{(u, v, w) : u < 0, -u\exp(v/u) \leq \exp(1)w\} \cup \{(0, v, w) : v, w \geq 0\}.$$

[Hint: argue that if $w > 0$ and $u < 0$, then $-u \exp v/u \leq \exp(1)w$ holds. You can use the fact (without proof) that minimizers of a differentiable convex function $f : \mathbb{R} \to \mathbb{R}$ occur at points where $f'(x) = 0$. We prove a more general version of this in lectures.]