MA1407.00050 Mathematical Algorithms II **SNU, Fall 2025**
Lecture #20: SDP-based Approximation Algorithms November 18, 2025
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1 Outline

In this lecture, we cover

- the 0.878 approximation algorithm for Max-Cut,
- an SDP-based approximation algorithm for binary convex quadratic maximization.

2 SDP-based Approximation Algorithm for Max Cut

In this section, we present a famous approximation algorithm for the max-cut problem by Goemans and Williamson [GW94] based on semidefinite programming (SDP).

Consider the following integer quadratic programming formulation of the max-cut problem:

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2}\sum_{ij\in E}(1-x_ix_j) \\ \text{subject to} \quad & x_i \in \{-1,1\}, \quad i=1,\dots,n. \end{aligned}$$

Here, each variable x_i indicates which set vertex i belongs to: if $x_i = 1$, then vertex i is in V_1; otherwise, it is in V_2. Note that for each edge $ij \in E$, it contributes 1 to the objective function if and only if $x_i \neq x_j$, i.e., the edge crosses the partition. Then take vector $x = (x_1, \dots, x_n)^\top \in \{-1,1\}^n$ and let $X = xx^\top$. It follows that X is a positive semidefinite matrix with rank 1 and $X_{ii} = 1$ for all i. Therefore, we may reformulate the max-cut problem as follows:

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2}\sum_{ij\in E}(1-X_{ij}) \\ \text{subject to} \quad & X_{ii} = 1, \quad i=1,\dots,n, \\ & X \succeq 0, \\ & \text{rank}(X) = 1. \end{aligned}$$

However, the rank constraint is non-convex. By dropping it, we obtain the following semidefinite programming (SDP) relaxation of the max-cut problem:

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2}\sum_{ij\in E}(1-X_{ij}) \\ \text{subject to} \quad & X_{ii} = 1, \quad i=1,\dots,n, \\ & X \succeq 0. \end{aligned}$$

Any SDP is a convex optimization problem that can be solved in polynomial time up to arbitrary accuracy. Let X^* be an optimal solution to the SDP relaxation. Since $X^* \succeq 0$, there exist vectors $y_1, \dots, y_n \in \mathbb{R}^n$ such that $X^*_{ij} = y_i^\top y_j$ for all i,j due to the Cholesky decomposition. Note that $\|y_i\|_2^2 = X^*_{ii} = 1$ for all i. The Goemans-Williamson algorithm is as follows:

Algorithm 1 Goemans-Williamson Algorithm for Max-Cut

Solve the SDP relaxation to obtain an optimal solution X^*.
Take a Cholesky decomposition of X^* given by $X^* = Y^\top Y$.
Take the columns $y_1, \dots, y_n$ of Y.
Sample a random vector g from the unit sphere $\{g \in \mathbb{R}^n : \|g\|_2 = 1\}$ uniformly at random.
Consider the hyperplane $\{y \in \mathbb{R}^n : g^\top y = 0\}$.
Partition the vertices into $V_1 = \{i : g^\top y_i \geq 0\}$ and $V_2 = \{i : g^\top y_i < 0\}$.

Theorem 20.1 ([GW94]). *Let $G = (V, E)$ be an undirected graph. The Goemans-Williamson algorithm (Algorithm 1) outputs a partition (V_1, V_2) such that the expected number of edges crossing the partition is at least 0.878 times the optimal value of the max-cut problem.*

Proof. Note that

$$\begin{aligned}
\mathbb{E}\left[|\delta(V_1, V_2)|\right] &= \sum_{ij \in E} \mathbb{P}\left[ij \in \delta(V_1, V_2)\right] \\
&= \sum_{ij \in E} \mathbb{P}\left[(g^\top y_i \geq 0, g^\top y_j < 0) \text{ or } (g^\top y_i < 0, g^\top y_j \geq 0)\right] \\
&= \sum_{ij \in E} \mathbb{P}\left[\text{the hyperplane separates } y_i \text{ and } y_j\right].
\end{aligned}$$

Here, for each edge $ij \in E$, let $\theta_{ij} \in [0, \pi]$ be the angle between y_i and y_j. It follows that

$$\mathbb{P}\left[\text{the hyperplane separates } y_i \text{ and } y_j\right] = \frac{\theta_{ij}}{\pi}.$$

Consequently, we have

$$\mathbb{E}\left[|\delta(V_1, V_2)|\right] = \sum_{ij \in E} \frac{\theta_{ij}}{\pi}.$$

On the other hand, since $\cos(\theta_{ij}) = y_i^\top y_j = X^*_{ij}$, we have

$$\mathrm{OPT} \leq \sum_{ij \in E} \frac{1}{2}(1 - X^*_{ij}) = \sum_{ij \in E} \frac{1}{2}(1 - \cos(\theta_{ij})).$$

It is known that for any $\theta \in [0, \pi]$,

$$\frac{\theta}{\pi} \geq 0.878 \cdot \frac{1}{2}(1 - \cos(\theta)).$$

Therefore, we obtain

$$\mathbb{E}\left[|\delta(V_1, V_2)|\right] = \sum_{ij \in E} \frac{\theta_{ij}}{\pi} \geq 0.878 \sum_{ij \in E} \frac{1}{2}(1 - \cos(\theta_{ij})) \geq 0.878 \cdot \mathrm{OPT},$$

as required. □

3 Binary Convex Quadratic Maximization

In this section, we consider the following binary convex quadratic maximization problem:

$$\begin{array}{ll} \text{maximize} & x^\top Q x \\ \text{subject to} & x \in \{-1,1\}^n, \end{array}$$

where $Q \in \mathbb{R}^{n\times n}$ is a positive semidefinite matrix. In fact, the max-cut problem is a special case of the binary convex quadratic maximization problem, as shown below. Let L denote the Laplacian matrix of the graph $G = (V, E)$ defined by

$$L_{ij} = \begin{cases} \deg(i), & \text{if } i = j, \\ -1, & \text{if } i \text{ and } j \text{ are adjacent}, \\ 0, & \text{otherwise}. \end{cases}$$

Then it follows that

$$\begin{aligned} x^\top L x &= \sum_{i=1}^n \deg(i) x_i^2 - \sum_{i=1}^n \sum j : ij \in E x_i x_j \\ &= \sum_{ij\in E} (x_i^2 + x_j^2) - 2 \sum_{ij\in E} x_i x_j \\ &= 2 \sum_{ij\in E} (1 - x_i x_j). \end{aligned}$$

Here, it is known that the Laplacian matrix L is positive semidefinite. Therefore, the max-cut problem can be formulated as the following binary convex quadratic maximization problem:

$$\begin{array}{ll} \text{maximize} & \dfrac{1}{4} x^\top L x \\ \text{subject to} & x \in \{-1,1\}^n. \end{array}$$

In the previous section, we have seen that the SDP-based algorithm due to Goemans and Williamson can achieve an approximation ratio of at least 0.878 for the max-cut problem. In this section, we will see that the same algorithm guarantees a constant approximation algorithm for binary convex quadratic maximization problems. As for the max-cut formulation, we deduce

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^n \sum_{j=1}^n Q_{ij} X_{ij} \\ \text{subject to} & X_{ii} = 1, \quad i = 1, \ldots, n, \\ & X \succeq 0, \end{array}$$

which is an SDP relaxation of the binary convex quadratic maximization problem. Let X^* be an optimal solution to the SDP relaxation. Then we may apply the same algorithm described as in Algorithm 2. Nesterov [Nes98] showed that Algorithm 2 achieves an approximation ratio of at least $2/\pi$ for binary convex quadratic maximization problems.

Theorem 20.2 ([Nes98])**.** *Let $Q \in \mathbb{R}^{n\times n}$ be a positive semidefinite matrix. The SDP-based algorithm for binary convex quadratic maximization (Algorithm 2) outputs a solution $x \in \{-1,1\}^n$ such that the expected objective value is at least $\frac{2}{\pi}$ times the optimal value of the binary convex quadratic maximization problem.*

Algorithm 2 SDP-based Algorithm for Binary Convex Quadratic Maximization

Solve the SDP relaxation to obtain an optimal solution X^*.
Take a Cholesky decomposition of X^* given by $X^* = Y^\top Y$.
Take the columns $y_1, \ldots, y_n$ of Y.
Sample a random vector g from the unit sphere $\{g \in \mathbb{R}^n : \|g\|_2 = 1\}$ uniformly at random.
Consider the hyperplane $\{y \in \mathbb{R}^n : g^\top y = 0\}$.
Set $x_i = \mathrm{sign}(g^\top y_i)$ for $i = 1, \ldots, n$.

Proof. We have argued in the proof of Theorem 20.1 that with the hyperplane $\{y \in \mathbb{R}^n : g^\top y = 0\}$,

$$\frac{1}{2}\left(1 - \mathbb{E}\left[x_i x_j\right]\right) = \mathbb{P}\left[\text{the hyperplane separates } y_i \text{ and } y_j\right] = \frac{\theta_{ij}}{\pi},$$

where $\theta_{ij} \in [0, \pi]$ is the angle between y_i and y_j. This implies that

$$\begin{aligned}
\mathbb{E}\left[x^\top Q x\right] &= \sum_{i=1}^n \sum_{j=1}^n Q_{ij} \mathbb{E}\left[x_i x_j\right] \\
&= \sum_{i=1}^n \sum_{j=1}^n Q_{ij} \left(1 - \frac{2}{\pi} \cdot \theta_{ij}\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n Q_{ij} \left(1 - \frac{2}{\pi} \cdot \arccos(y_i^\top y_j)\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n Q_{ij} \left(1 - \frac{2}{\pi} \cdot \arccos(X^*_{ij})\right) \\
&= \frac{2}{\pi} \sum_{i=1}^n \sum_{j=1}^n Q_{ij} \cdot \arcsin(X^*_{ij})
\end{aligned}$$

It is known that for any positive semidefinite matrix X, the matrix $\arcsin(X) - X$ is also positive semidefinite. Then, since Q is positive semidefinite, we have

$$\sum_{i=1}^n \sum_{j=1}^n Q_{ij} \cdot \arcsin(X^*_{ij}) \geq \sum_{i=1}^n \sum_{j=1}^n Q_{ij} X^*_{ij} \geq \mathrm{OPT},$$

where OPT is the optimal value of the binary convex quadratic maximization problem. Therefore, we obtain

$$\mathbb{E}\left[x^\top Q x\right] \geq \frac{2}{\pi} \cdot \mathrm{OPT},$$

as required. □

References

[GW94] Michel X. Goemans and David P. Williamson. .879-approximation algorithms for max cut and max 2sat. In *Proceedings of the Twenty-Sixth Annual ACM Symposium on Theory of Computing*, STOC '94, page 422–431, New York, NY, USA, 1994. Association for Computing Machinery. 2, 20.1

[Nes98] Yu Nesterov. Semidefinite relaxation and nonconvex quadratic optimization. *Optimization Methods and Software*, 9(1-3):141–160, 1998. 3, 20.2