

MA1407.000500 Mathematical Algorithms II Assignment 2

Fall 2025

Out: 2nd October 2025

Due: 14th October 2025 at 11:59pm

Instructions

- Submit a PDF document with your solutions through the assignment portal on KLMS by the due date. Please ensure that your name and student ID are on the front page.
- Late assignments will **not** be accepted except in extenuating circumstances. Special consideration should be applied for in this case.
- It is **required** that you typeset your solutions in LaTeX. Handwritten solutions will not be accepted.
- Spend some time ensuring your arguments are **coherent** and your solutions **clearly** communicate your ideas.

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100

1. Recall that for a square symmetric matrix $A \in \mathbb{R}^d$, the Rayleigh quotient is defined as

$$R(A, x) = \frac{x^\top A x}{x^\top x} \quad \text{for any nonzero } x.$$

- (a) (10 points) Let $\lambda_{\max}$ denote the largest eigenvalue of A , and let $v_{\max}$ denote the associated eigenvector. Prove that $\max_{x \in \mathbb{R}^d: x \neq 0} R(A, x) = \lambda_{\max}$ and $\arg \max_{x \in \mathbb{R}^d: x \neq 0} R(A, x) = v_{\max}$.
- (b) (10 points) Let $\lambda_{\min}$ denote the smallest eigenvalue of A , and let $v_{\min}$ denote the associated eigenvector. Prove that $\min_{x \in \mathbb{R}^d: x \neq 0} R(A, x) = \lambda_{\min}$ and $\arg \min_{x \in \mathbb{R}^d: x \neq 0} R(A, x) = v_{\min}$.
2. (20 points) Let $A \in \mathbb{R}^d$ be a symmetric matrix that is not necessarily positive semidefinite. Suppose that the eigenvalues $\lambda_1, \dots, \lambda_d$ of A satisfy

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_d|.$$

Let v_i denote the eigenvector of A associated with λ_i for $i \in [d]$. Moreover, let x_t denote the iterate generated by the power methods after t iterations where the initial point is x_0 . Prove that the power method guarantees that for any $\epsilon > 0$, if

$$t \geq \frac{|\lambda_1|}{2(|\lambda_1| - |\lambda_2|)} \log \left(\frac{1}{\epsilon(v_1^\top x_0)^2} \right),$$

then

$$(v_1^\top x_t)^2 \geq 1 - \epsilon.$$

3. (20 points) Prove that the unit nuclear norm ball is equivalent to the convex hull of rank 1 matrices, i.e.,

$$\{X \in \mathbb{R}^{n \times p} : \|X\|_* \leq 1\} = \text{conv} \{uv^\top : \|u\|_2 = \|v\|_2 = 1, u \in \mathbb{R}^n, v \in \mathbb{R}^p\}.$$

4. (20 points) Let $A \in \mathbb{R}^{n \times p}$ be an $n \times p$ matrix. Let u and v be the top left and right singular vectors of $-A$, respectively. Prove that

$$k \cdot uv^\top \in \arg \min \{A^\top X : \|X\|_* \leq k\}.$$

5. (20 points) Suppose that a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is β -smooth in the ℓ_2 -norm. Then we apply gradient descent with an initial point x_1 with step size

$$\eta_t = C \left(\sum_{s=1}^t \|\nabla f(x_s)\|_2^2 \right)^{-1/2}$$

for $t \geq 1$ where $C \leq \|\nabla f(x_1)\|_2 / \beta$. Let $x_2, \dots, x_T$ denote the iterates generated by the gradient descent after $T - 1$ iterations. Prove that

$$\min \{\|\nabla f(x_s)\|_2^2 : 1 \leq s \leq t\} \leq \frac{4(f(x_1) - f(x^*))^2}{C^2 T}$$

where $x^* \in \arg \min_{x \in \mathbb{R}^d} f(x)$.