Lecturer: Dabeen Lee

1 Outline

In this lecture, we study

- Fourier-Motzkin elimination,
- Farkas' lemma,
- Linear, convex, conic, affine combinations,
- Minkowski-Weyl theorem for cones.

2 Fourier-Motzkin elimination

We learned how to test the feasibility of $Ax = b$ with $x \in \mathbb{Z}^d$. The essential idea was to convert the equality system $Ax = b$ to another system with the Hermite normal form of A. Then checking the integrality of a solution boils down to simply enumerating and checking some components of a vector. In this section, we study how to check the feasibility of a system of linear inequalities:

$$Ax \leq b, \quad x \in \mathbb{R}^d.$$

Here, x contains continuous variables, so the feasibility problem is relevant to linear programming. Writing out the matrix inequality into linear inequalities, we have

$$\sum_{j=1}^{d-1} a_{ij}x_j + a_{id}x_d \leq b_i, \quad i = 1, \ldots, m$$

where we separate out the last variable from the sum. We will use the **Fourier-Motzkin elimination** method to eliminate variable x_d from the inequalities. Basically, the idea is that if $Ax \leq b$ has a feasible solution, then the system obtained after eliminating one variable by Fourier-Motzkin elimination would also have a solution. The resulting system has one less variable than the original system. As we continue this procedure, we would obtain an inequality system with no variable, such as $0 \leq 2$, we can check whose feasibility immediately.

Before we explain the method in general, let us consider a small example.

Example 9.1. Consider the following system of linear inequalities

$$\begin{array}{ccccc} -x_1 & -x_2 & & \leq 2 \\ x_1 & & -x_3 & \leq 0 \\ & x_2 & -x_3 & \leq 0 \\ x_1 & +x_2 & +x_3 & \leq 4 \end{array}$$

By adding up the second and the last inequalities, we obtain $2x_1 + x_2 \leq 4$, and by adding up the third and the last inequalities, we obtain $x_1 + 2x_2 \leq 4$. Hence, we deduce

$$\begin{array}{ccc} -x_1 & -x_2 & \leq 2 \\ 2x_1 & +x_2 & \leq 4 \\ x_1 & +2x_2 & \leq 4 \end{array}$$

that does not contain variable x_3.

The basic idea is Fourier-Motzkin elimination is aggregating inequalities.

1. Fix a variable to eliminate.

2. Take an inequality with a positive coefficient of the variable and an inequality with a negative coefficient.

3. Deduce an inequality by adding up the two inequalities.

Let (I^0, I^+, I^-) be a partition of $[m]$ defined as follows.

$$\begin{aligned} I^0 &= \{i \in [m] : \ a_{id} = 0\}, \\ I^+ &= \{i \in [m] : \ a_{id} > 0\}, \\ I^- &= \{i \in [m] : \ a_{id} < 0\}. \end{aligned}$$

Let $i_1 \in I^+$ and $i_2 \in I^-$. Then we deduce the following two inequalities.

$$\begin{aligned} \sum_{j=1}^{d-1} \frac{a_{i_1 j}}{a_{i_1 d}} x_j + x_d &\leq \frac{b_{i_1}}{a_{i_1 d}} \\ \sum_{j=1}^{d-1} \frac{a_{i_2 j}}{-a_{i_2 d}} x_j - x_d &\leq \frac{b_{i_2}}{-a_{i_2 d}}. \end{aligned}$$

Adding up these two inequalities, we obtain

$$\sum_{j=1}^{d-1} \left(\frac{a_{i_1 j}}{a_{i_1 d}} - \frac{a_{i_2 j}}{a_{i_2 d}} \right) x_j \leq \frac{b_{i_1}}{a_{i_1 d}} - \frac{b_{i_2}}{a_{i_2 d}}.$$

Applying this procedure for every pair of $i_1 \in I^+$ and $i_2 \in I^-$, we deduce the following system of linear inequalities.

$$\begin{aligned} \sum_{j=1}^{d-1} a_{ij} x_j \leq b_i, \quad & i \in I^0, \\ \sum_{j=1}^{d-1} \left(\frac{a_{i_1 j}}{a_{i_1 d}} - \frac{a_{i_2 j}}{a_{i_2 d}} \right) x_j \leq \frac{b_{i_1}}{a_{i_1 d}} - \frac{b_{i_2}}{a_{i_2 d}}, \quad & i_1 \in I^+, i_2 \in I^-. \end{aligned} \tag{9.1}$$

Theorem 9.2. *$(\bar{x}_1, \ldots, \bar{x}_{d-1})$ satisfies (9.1) if and only if $(\bar{x}_1, \ldots, \bar{x}_{d-1}, \bar{x}_d)$ satisfies $Ax \leq b$ for some $\bar{x}_d$. Hence, the system (9.1) has a feasible solution if and only if $Ax \leq b$ has a feasible solution.*

Proof. **($\Leftarrow$)** We showed that if $Ax \leq b$ holds, then (9.1) holds. Hence, if $\bar{x}$ satisfies $Ax \leq b$, then $\tilde{x} := (\bar{x}_1, \ldots, \bar{x}_{d-1})$ satisfies (9.1), in which case $\tilde{x}$ is a feasible solution to (9.1).

($\Rightarrow$) Let $\tilde{x} := (\bar{x}_1, \ldots, \bar{x}_{d-1})$ be a solution satisfying (9.1). Then for every pair of $i_1 \in I^+$ and $i_2 \in I^-$, we have

$$\frac{b_{i_2}}{a_{i_2 d}} - \sum_{j=1}^{d-1} \frac{a_{i_2 j}}{a_{i_2 d}} \bar{x}_j \leq \frac{b_{i_1}}{a_{i_1 d}} - \sum_{j=1}^{d-1} \frac{a_{i_1 j}}{a_{i_1 d}} \bar{x}_j.$$

In particular,

$$\max_{i_2\in I^-}\left\{\frac{b_{i_2}}{a_{i_2d}}-\sum_{j=1}^{d-1}\frac{a_{i_2j}}{a_{i_2d}}\bar{x}_j\right\}\le\min_{i_1\in I^+}\left\{\frac{b_{i_1}}{a_{i_1d}}-\sum_{j=1}^{d-1}\frac{a_{i_1j}}{a_{i_1d}}\bar{x}_j\right\}.$$

Let us choose $\bar{x}_d$ between the left-hand side value and the right-hand side value, i.e.,

$$\max_{i_2\in I^-}\left\{\frac{b_{i_2}}{a_{i_2d}}-\sum_{j=1}^{d-1}\frac{a_{i_2j}}{a_{i_2d}}\bar{x}_j\right\}\le\bar{x}_d\le\min_{i_1\in I^+}\left\{\frac{b_{i_1}}{a_{i_1d}}-\sum_{j=1}^{d-1}\frac{a_{i_1j}}{a_{i_1d}}\bar{x}_j\right\}.$$

In this case,

$$a_{i_1d}\bar{x}_d\le b_{i_1}-\sum_{j=1}^{d-1}a_{i_1j}\bar{x}_j,\quad i_1\in I^+$$

$$b_{i_2}-\sum_{j=1}^{d-1}a_{i_2j}\bar{x}_j\ge a_{i_2d}\bar{x}_d,\quad i_2\in I^-$$

Therefore, $\bar{x}=(\bar{x}_1,\ldots,\bar{x}_{d-1},\bar{x}_d)$ satisfies the system $Ax\le b$, as required. □

Algorithm 1 Fourier-Motzkin elimination procedure

Input : A, b.
$A^d\leftarrow A$, $b^d\leftarrow b$.
Eliminate variable x_d as above to get $A^{d-1}x\le b^{d-1}$ where the column for variable x_d is 0.
Continue until $A^0x\le b^0$ where $A^0=0$.

Corollary 9.3. *System $Ax\le b$ has a feasible solution if and only if $b^0\ge 0$ where b^0 is given in Algorithm 1.*

With the Fourier-Motzkin elimination method, we can prove the following famous result of Farkas on checking the feasibility of linear system.

Theorem 9.4 (Farkas' lemma)**.** *System $Ax\le b$ is infeasible if and only if the system $\lambda^\top A=0$, $\lambda^\top b<0$, and $\lambda\ge 0$ is feasible.*

Proof. **(⇐)** Suppose that $Ax\le b$ is feasible for a contradiction. As $\lambda\ge 0$, we have

$$\lambda^\top Ax\le\lambda^\top b.$$

Moreover, as $\lambda^\top A=0$ and $\lambda^\top b<0$, we deduce that

$$0=\lambda^\top Ax\le\lambda^\top b<0,$$

a contradiction. Therefore, $Ax\le b$ is infeasible.

(⇒) Assume that $Ax\le b$ is infeasible. By Corollary 9.3, applying Fourier-Motzkin elimination results in $0\le b^0$ which is infeasible. Then $b_i^0<0$ for some component i. Notice that what Fourier-Motzkin elimination does is to multiply inequalities by some positive numbers and add up the resulting inequalities. Hence, the Fourier-Motzkin elimination procedure can be mimicked by some nonnegative multiplier vector $\lambda\ge 0$ so that $\lambda^\top A=0$ and $\lambda^\top b=b_i^0$. As b_i^0, we have $\lambda^\top b<0$, as required. □

3 Linear, convex, conic, and affine combinations

Let $v^1, \dots, v^k \in \mathbb{R}^d$ be d-dimensional vectors. A **linear combination** of the vectors is

$$\sum_{i=1}^{k} \alpha_k v^k$$

for some $\alpha_1, \dots, \alpha_k \in \mathbb{R}$. We say that vectors $v^1, \dots, v^k$ are **linearly independent** if $\sum_{i=1}^{k} \alpha_k v^k = 0$ has a unique solution $\alpha_1 = \cdots = \alpha_k = 0$. Otherwise, we say that the vectors are **linearly dependent**.

We call $V \subseteq \mathbb{R}^d$ a **linear subspace** if V is closed under taking linear combinations. The **dimension** of a linear subspace V is defined as the maximum number of linearly independent vectors in V. A **basis** of a linear subspace V is a maximal set of linearly independent vectors in V.

A linear combination $\lambda_1 v^1 + \cdots + \lambda_k v^k$ of vectors $v^1, \dots, v^k \in \mathbb{R}^d$ is a **convex combination** if

$$\sum_{i=1}^{k} \lambda_i = 1 \quad \text{and} \quad \lambda_i \geq 0 \text{ for } i = 1, \dots, k.$$

A linear combination $\beta_1 v^1 + \cdots + \beta_k v^k$ of vectors $v^1, \dots, v^k \in \mathbb{R}^d$ is a **conic combination** if

$$\beta_1, \dots, \beta_k \geq 0.$$

In other words, any nonnegative linear combination is a conic combination. A set $C \subseteq \mathbb{R}^d$ is a **cone** if for any $v \in C$ and $\alpha > 0$, we have $\alpha v \in C$. Furthermore, if a cone C contains every conic combination of vectors in C, then it is called a **convex cone**. The *conic hull* of a set X, denoted $\text{cone}(X)$, is the set of all conic combinations of points in X. By definition,

$$\text{cone}(X) = \left\{ \sum_{i=1}^{n} \lambda_i v^i : \begin{array}{l} n \in \mathbb{N},\ v^1, \dots, v^n \in X, \\ \beta_1, \dots, \beta_n \geq 0 \end{array} \right\}.$$

As $\text{conv}(X)$, $\text{cone}(X)$ is always convex. Figure 9.1 shows an example taking the conic hull of a set in $\mathbb{R}^2$. A linear combination $\theta_1 v^1 + \cdots + \theta_k v^k$ of vectors $v^1, \dots, v^k \in \mathbb{R}^d$ is a **affine combination**

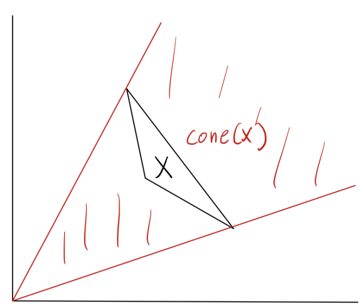

Figure 9.1: Taking the conic hull of a triangle in $\mathbb{R}^2$

if

$$\theta_1 + \cdots + \theta_k = 1.$$

In contrast to covex combinations, affine combinations allow negative multipliers. The **affine hull** of a set X is the set of all affine combinations of points in X. The affine hull of X is also referred to as the **affine subspace** spanned by X.

We say that vectors $v^1, \dots, v^k$ are **affinely independent** if

$$\sum_{i=1}^{k} \theta_i v^i = 0, \quad \sum_{i=1}^{k} \theta_i = 0$$

has a unique solution $\theta_1 = \cdots = \theta_k = 0$. The **dimension** of any set S, denoted $\dim(S)$, is defined as the maximum number of affinely independent vectors in S minus 1.

In Figure 9.2, we have a set S of two points in $\mathbb{R}^2$. The red line segment is $\mathrm{conv}(S)$, the green line through the two points is the affine subspace spanned by S, the blue cone depicts $\mathrm{cone}(S)$, and lastly, the orange regin (in fact, $\mathbb{R}^2$) is the linear subspace spanned by S.

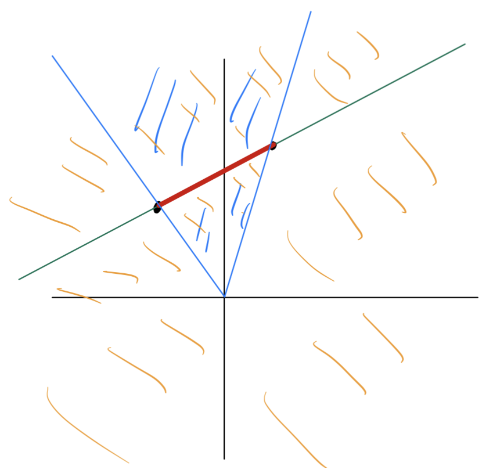

Figure 9.2: Comparing the linear subspace, the affine subspace, the convex hull, and the conic hull

Theorem 9.5. *An affine subspace is a translation of a linear subspace. For an affine subspace $V \subseteq \mathbb{R}^d$, there exist matrices A and b such that $V = \{x \in \mathbb{R}^d : Ax = b\}$.*

4 Minkowski-Weyl theorem for cones

A set $C \subseteq \mathbb{R}^d$ is a **polyhedral cone** if it is defined by a **finite** number of half-spaces whose boundaries go through the origin, i.e,

$$C = \{x \in \mathbb{R}^d : \; Ax \leq 0\}.$$

Theorem 9.6 (Minkowski-Weyl theorem for cones)**.** *A set $C \subseteq \mathbb{R}^d$ is a polyhedral cone if and only if*

$$C = \mathrm{cone}(r^1, \dots, r^k)$$

for some vectors $r^1, \dots, r^k$.

Proof. We prove direction ($\Leftarrow$) using Fourier-Motzkin elimination. For the other direction, we refer to the book.

As C is the conic hull of $r^1, \ldots, r^k$, we have

$$C = \left\{ x \in \mathbb{R}^d : \; \exists \mu \geq 0 \text{ s.t. } x = \sum_{i=1}^{k} \mu_i r^i \right\}.$$

Let R be the $d \times k$ matrix whose columns are $r^1, \ldots, r^k$. Then C can be written as

$$C = \left\{ x \in \mathbb{R}^d : \; \exists \mu \geq 0 \text{ s.t. } x = R\mu \right\}.$$

Then C is defined by the system

$$x - R\mu = 0, \quad \mu \geq 0.$$

By applying Fourier-Motzkin elimination, we may eliminate variables μ and deduce system $Ax \leq b$. Then it follows from Theorem 9.2 that

$$C = \left\{ x \in \mathbb{R}^d : \; Ax \leq b \right\}.$$

Here, the original system given by $x - R\mu = 0$ and $\mu \geq 0$ has all its right-hand sides 0. Then any sytem obtained after Fourier-Motzkin elimination also has right-hand sides 0. Therefore, $b = 0$ and C is defined by $Ax \leq 0$. Therefore, C is a polyhedral cone. □

Theorem 9.6 has the following immediate consequences.

- Given a matrix A, there exists a finite set of vectors $r^1, \ldots, r^k$ such that $\{x \in \mathbb{R}^d : Ax \leq 0\} = \text{cone}(r^1, \ldots, r^k)$.
- Given a finite set of vectors $r^1, \ldots, r^k$, there exists a matrix A such that $\text{cone}(r^1, \ldots, r^k) = \{x \in \mathbb{R}^d : Ax \leq 0\}$.