1 Outline

In this lecture, we study

- inventory planning and mixing set,
- chance-constrained programs,
- union of polytopes.

2 Inventory planning: Mixing set

We consider a two-stage inventory planning problem. A retail store prepares some inventory of items before the market opens, and the retail store can observe the actual demand after the market opens. If the prepared amount of items is not enough for satisfying the demand, then the retail store can urgently secure more items at a higher cost.

- y: the amount of items that the retail store prepares before the market opens.
- h: the unit cost of preparing items before the market opens.
- b: the stochastic demand for items.
- c: the unit cost of securing more items after the market opens.

Assumption 1. Before the market opens, we may prepare a fractional quantity, i.e., y can be fractional. However, securing items after the market opens is more restrictive, and the extra order should be of an integer quantity.

Given that the inventory of items is y and the demand is b, we define $f(y,b)$ as the minimum amount of extra orders after the market opens. Then $f(y,b)$ is given by

$$f(y,b) = \min\{z : \ y + z \geq b, \ z \in \mathbb{Z}_+\}.$$

The problem is to decide the order quantities before and after the market opens so as to satisfy the market demand while minimizing the total cost. The problem can be modeled by

$$\min_{y} \quad hy + c \cdot \mathbb{E}_b\left[f(y,b)\right].$$

Assumption 2. There are n possibilities, given by $b_1, \ldots, b_n$, for the stochastic demand b. Historically, the demand is equal to value b_i with probability p_i, i.e.,

$$\mathbb{P}\left[b = b_i\right] = p_i.$$

Here, $p_1, \ldots, p_n \geq 0$ and $\sum_{i=1}^n p_i = 1$. We assume that the probability distribution is known to the decision-maker.

Based on this assumption, the problem is equivalent to

$$\min_{y} \quad hy + c\sum_{i=1}^{n} p_i f(y, b_i).$$

As $f(y, b_i)$ itself is defined by an optimization problem, the problem can be written as

$$\min_{y} \quad hy + c\sum_{i=1}^{n} p_i \cdot \min\{z : \ y + z \geq b_i, \ z \in \mathbb{Z}_+\}.$$

This type of problem is called a **two-stage optimization** model. Here, y is called the **first-stage decision variable**, and z is called the **second-stage decision variable**. Moreover, each case of demand realization is called a **scenario**. There are n scenarios, and scenario i occurs with probability p_i.

In fact, we may reformulate the two-stage optimization model as a single optimization problem as follows. For each scenario, we use variable x_i to replace z. Then we deduce

$$\min_{y} \quad hy + c\sum_{i=1}^{n} p_i \cdot \min\{x_i : \ y + x_i \geq b_i, \ x_i \in \mathbb{Z}_+\}.$$

Note that the inner optimization problem is also a minimization problem. Then we may optimize over both the first-stage and second-stage variables simultaneously. To be specific, the following is an equivalent reformulation of the problem.

$$\begin{aligned} \min \quad & hy + c\sum_{i=1}^{n} p_i x_i \\ \text{s.t.} \quad & y + x_i \geq b_i, \quad i = 1, \dots, n, \\ & y \in \mathbb{R}_+, \ x \in \mathbb{Z}_+^n. \end{aligned}$$

The solution set of this model

$$\left\{(y, x) \in \mathbb{R}_+ \times \mathbb{Z}_+^n : \ y + x_i \geq b_i, \quad i = 1, \dots, n\right\}$$

is called the **mixing set** [GP01]. The convex hull of the mixing set is well-understood.

Theorem 8.1 ([GP01])**.** *The convex hull of the mixing set is described by the* ***mixing inequalities****. Although the number of mixing inequalities is exponential in n, we may separate a violated mixing inequality in polynomial time.*

3 Chance-constrained programs

Let us consider the previous inventory planning setting again. In the previous section, we allowed ordering more items after the market opens. However, in this section, we assume that no purchase can be made after the market opens. Therefore, the decision-maker has to prepare enough quantity of items before the market opens, based on the distribution of the stochastic demand.

The first attempt is to prepare again all possible scenarios. Basically, we target the largest possible demand by solving

$$\begin{aligned} \min \quad & hy \\ \text{s.t.} \quad & y \geq b_i, \quad i = 1, \dots, n, \\ & y \in \mathbb{R}_+. \end{aligned}$$

However, targeting the largest possible demand may be a too conservative decision. Maybe the largest possible demand value occurs with probability less than 0.1% while we would face a moderate demand level with proability in most cases. How do we take this into account? Let us consider

$$\begin{aligned} \min \quad & hy \\ \text{s.t.} \quad & \mathbb{P}\left[y \geq b\right] \geq 0.95 \\ & y \in \mathbb{R}_+. \end{aligned}$$

This optimization model is called a **chance-constrained program**. Note that the constraint requires that we satisfy the stochastic demand with at least 95% chance. We might not satisfy the demand in some cases, but as long as the failure probability is at most 5%, we hare happy.

In fact, the chance-constrained program can be reformulated as an integer program. Note that

$$\mathbb{P}\left[y \geq b\right] \geq 0.95$$

is equivalent to

$$\mathbb{P}\left[y < b\right] \leq 0.05.$$

Moreover,

$$\mathbb{P}\left[y < b\right] = \sum_{i=1}^{n} p_i \cdot \mathbf{1}\left[y < b_i\right]$$

where

$$\mathbf{1}\left[y < b_i\right] = \begin{cases} 1, & \text{if } y < b_i, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$z_i = \begin{cases} 0, & \text{if the demand for scenario } i \text{ is satisfied,} \\ 1, & \text{otherwise.} \end{cases}$$

Basically, we use the binary variable z_i to model the indicator function $\mathbf{1}\left[y < b_i\right]$. Then the chance-constrained program can be reformulated as the following integer program.

$$\begin{aligned} \min \quad & hy \\ \text{s.t.} \quad & y + b_i z_i \geq b_i, \quad i = 1, \ldots, n, \\ & \sum_{i=1}^{n} p_i z_i \leq 0.05, \\ & y \in \mathbb{R}_+, z \in \{0,1\}^n. \end{aligned}$$

The solution set of this model

$$\left\{(y, x) \in \mathbb{R}_+ \times \mathbb{Z}_+^n : \ y + x_i \geq b_i, \quad i = 1, \ldots, n\right\}$$

is called the **binary mixing set** [LAN10]. The convex hull of the mixing set is also well-understood.

Theorem 8.2 ([ANS00, LAN10]). *The convex hull of the binary mixing set is described by the* ***mixing inequalities****. Although the number of mixing inequalities is exponential in n, we may separate a violated mixing inequality in $O(n \log n)$ time.*

4 Union of polytopes

Consider the inventory planning problem again. What if the second-stage order quantity cannot be an arbitrary integer but can be chosen from some fixed list of options? Suppose that the second-stage order quantity z satisfies

$$z \in \{q_1, \ldots, q_\ell\}.$$

Moreover, assume that the demand is fixed and known to be b. In this case, the corresponding optimization model would be

$$\begin{aligned} \min \quad & hy + cz \\ \text{s.t.} \quad & y + z \geq b \\ & y \in \mathbb{R}_+, \ z \in \{q_1, \ldots, q_\ell\}. \end{aligned}$$

Note that the feasible region is the union of ℓ sets given by

$$Q_j = \{(y, z) \in \mathbb{R} \times \mathbb{R} : \ y + z \geq b, \ y \geq 0, \ z = q_j\}.$$

In other words, the optimization model is equivalent to

$$\begin{aligned} \min \quad & hy + cz \\ \text{s.t.} \quad & (y, z) \in \bigcup_{j=1}^{\ell} Q_j. \end{aligned}$$

Here, since $hy + cz$ is linear and therefore convex, the optimization problem is equivalent to

$$\begin{aligned} \min \quad & hy + cz \\ \text{s.t.} \quad & (y, z) \in \operatorname{conv}\left(\bigcup_{j=1}^{\ell} Q_j\right). \end{aligned}$$

Note that each Q_j is a polyhedron. How do we obtain the convex hull of the ℓ polyhedra?

In general, consider k polyhedra given by

$$P_i = \left\{x \in \mathbb{R}^d : \ A^i x \leq b^i\right\}$$

for $i = 1, \ldots, k$.

Assumption 3. $P_1, \ldots, P_k$ are all bounded and nonempty.

Note that we can make each Q_j bounded, because we know that the optimal value of y is at most b. Hence, the question is as to how we model

$$\bigcup_{i=1}^{k} P_i \quad \text{and} \quad \operatorname{conv}\left(\bigcup_{i=1}^{k} P_i\right).$$

Since $P_1, \ldots, P_k$ are bounded, there exists a large constant M such that

$$A^i x \leq b^i + M\mathbf{1}$$

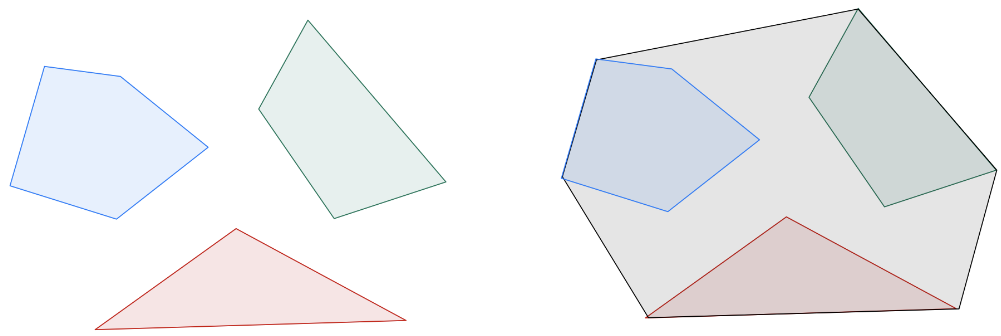

Figure 8.1: Union of polytopes and its convex hull

is always satisfied for all $x \in \mathbb{R}^d$ and $i \in [k]$. Consider the following formulation.

$$\begin{aligned} A^i x &\le b^i + M\mathbf{1}(1 - z_i), \quad i = 1, \dots, k \\ \sum_{i=1}^{k} z_i &= 1, \\ z &\in \{0,1\}^k, \\ x &\in \mathbb{R}^d. \end{aligned} \tag{8.1}$$

Theorem 8.3. *The set of vectors x satisfies the constraints in (8.1) with some z is the union of $P_1, \dots, P_k$.*

This is called a **big-M based formulation**. Note that we would to use a large number for M, which makes the corresponding LP relaxation weak. Instead of this formulation, we take the following formulation.

$$\begin{aligned} A^i x^i &\le b^i z_i, \quad i = 1, \dots, k \\ \sum_{i=1}^{k} z_i &= 1, \\ \sum_{i=1}^{k} x^i &= x, \\ z &\in \{0,1\}^k. \end{aligned} \tag{8.2}$$

Theorem 8.4. *The set of vectors x satisfies the constraints in (8.2) with some z is the union of $P_1, \dots, P_k$.*

It turns out that the second formulation is the tighest possible. Its relaxation is

$$\begin{aligned} A^i x^i &\le b^i z_i, \quad i = 1, \dots, k \\ \sum_{i=1}^{k} z_i &= 1, \\ \sum_{i=1}^{k} x^i &= x, \\ z &\in [0,1]^k. \end{aligned} \tag{8.3}$$

Theorem 8.5 ([Bal74]). *The set of vectors x satisfies the constraints in (8.3) with some z is*

$$\operatorname{conv}\left(\bigcup_{i=1}^{k} P_i\right),$$

the convex hull of the union of $P_1, \ldots, P_k$.

Proof. By the previous theorem, we have

$$\bigcup_{i=1}^{k} P_i = \left\{x : \exists (x^1, \ldots, x^k, z_1, \ldots, z_k) \text{ such that } (x, x^1, \ldots, x^k, z_1, \ldots, z_k) \text{ satisfies (8.2)}\right\}.$$

As (8.3) is a continuous relaxation of (8.2), it follows that

$$\bigcup_{i=1}^{k} P_i \subseteq \left\{x : \exists (x^1, \ldots, x^k, z_1, \ldots, z_k) \text{ such that } (x, x^1, \ldots, x^k, z_1, \ldots, z_k) \text{ satisfies (8.3)}\right\}.$$

Here, the set on the right-hand side is convex, so

$$\operatorname{conv}\left(\bigcup_{i=1}^{k} P_i\right) \subseteq \left\{x : \exists (x^1, \ldots, x^k, z_1, \ldots, z_k) \text{ such that } (x, x^1, \ldots, x^k, z_1, \ldots, z_k) \text{ satisfies (8.3)}\right\}.$$

Let x be a vector satisfying (8.3) together with some $(x^1, \ldots, x^k, z_1, \ldots, z_k)$. Note that

$$(x, x^1, \ldots, x^k, z_1, \ldots, z_k) = \sum_{i : z_i \neq 0} z_i \left(\frac{x^i}{z_i}, \underbrace{0, \ldots, 0, \frac{x^i}{z_i}, 0, \ldots, 0}_{i\text{th entry is nonzero}}, \underbrace{0, \ldots, 0, 1, 0, \ldots, 0}_{i\text{th entry is nonzero}}\right)$$

is a convex combination of points satisfying (8.2). This means that

$$x = \sum_{i : z_i \neq 0} z_i \cdot \frac{x^i}{z_i} \in \operatorname{conv}\left(\bigcup_{i=1}^{k} P_i\right).$$

Therefore,

$$\operatorname{conv}\left(\bigcup_{i=1}^{k} P_i\right) \supseteq \left\{x : \exists (x^1, \ldots, x^k, z_1, \ldots, z_k) \text{ such that } (x, x^1, \ldots, x^k, z_1, \ldots, z_k) \text{ satisfies (8.3)}\right\},$$

as required. □

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