Lecturer: Dabeen Lee

1 Outline

In this lecture, we study

- convex hull and reduction to linear programming,
- characterizing the convex hull of a two variable mixed integer linear set,
- methods for solving integer programming.

2 Solving a system of equations with integer constraints

Consider the following example.

Example 6.1. Solve the following system.

$$\begin{aligned} 10x_1 + 4x_2 + 3x_3 &= 3, \\ 58x_1 + 24x_2 + 19x_3 + 2x_4 &= 5, \\ 3x_1 + 2x_2 &= 5, \\ x_i &\in \mathbb{Z} \quad \text{for } i = 1, \ldots, 4. \end{aligned}$$

Then our constraint matrix A is

$$A = \begin{bmatrix} 10 & 4 & 3 & 0 \\ 58 & 24 & 19 & 2 \\ 3 & 2 & 0 & 0 \end{bmatrix}.$$

Eventually, we will get

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -2 & 0 & 1 & 6 \\ 3 & 0 & -1 & -9 \\ 3 & 0 & -2 & -8 \\ -6 & 1 & 2 & 10 \end{bmatrix}.$$

In this case, we get

$$U^{-1}x = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 0 \end{bmatrix} + k \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

from $HU^{-1}x = b$ and

$$x = \begin{bmatrix} -1 \\ 4 \\ -1 \\ -7 \end{bmatrix} + k \begin{bmatrix} 6 \\ -9 \\ -8 \\ 10 \end{bmatrix}$$

from $x = U(U^{-1}x)$.

2.1 Theorems of the alternative

The following result regards the solvability of a system of linear equations. The result is similar to Farkas' lemma for linear programming.

Theorem 6.2 (Fredholm Alternative)**.** *A system of m linear equations $Ax = b$ is infeasible if and only if there exists a vector $u \in \mathbb{R}^m$ such that $u^\top A = 0$ and $u^\top b \neq 0$.*

When the variables are restricted to be integers, whether a system of linear equations has a solution can be determined by a similar characterization.

Theorem 6.3 (Integer Farkas Lemma)**.** *Let $A \in \mathbb{Q}^{m\times d}$ and $b \in \mathbb{Q}^m$. The system $Ax = b,\ x \in \mathbb{Z}^d$ is infeasible if and only if there exists a vector u such that $A^\top u \in \mathbb{Z}^d$ and $u^\top b \notin \mathbb{Z}$.*

Proof. Suppose that $Ax = b,\ x \in \mathbb{Z}^d$ is feasible. Then $\forall u \in \mathbb{R}^m$ with $A^\top u \in \mathbb{Z}^d$, we have $u^\top b = u^\top Ax \in \mathbb{Z}$.

Suppose that $Ax = b,\ x \in \mathbb{Z}^d$ is infeasible. If $Ax = b$ is infeasible (even without the integrality constraint), there exists a vector $u \in \mathbb{R}^m$ such that $u^\top A = 0$ and $u^\top b \neq 0$ by the Fredholm Alternative. In this case, we can scale u so that $u^\top b$ is not an integer. To complete the proof, let us consider the case when $Ax = b$ is feasible and A has full row rank. Then A can be brought to its Hermite normal form $(D, 0) = AU$ for some unimodular matrix U. We saw that $Ax = b,\ x \in \mathbb{Z}^d$ has a solution if and only if $\bar{y} = D^{-1}b \in \mathbb{Z}^m$. Thus, $\exists i$ such that $\bar{y}_i \notin \mathbb{Z}$. Let $u^\top$ be the ith row of D^{-1}. Then $u^\top b$ is not an integer. Now, we need to show that $u^\top A$ is an integral vector. Note that

$$u^\top A = u^\top \begin{bmatrix} D & 0 \end{bmatrix} U^{-1} = e_i^\top U^{-1}.$$

where e_i denotes the ith unit vector. Since U is unimodular and has integer entries, U^{-1} also has integer entries. Therefore, $u^\top A \in \mathbb{Z}^d$. □

3 The knapsack problem

We are given d items. Assume that item $i \in [d]$ has weight w_i and value p_i. Moreover, the knapsack capacity is given by B. The problem is to choose a combination of items whose total weight is under the knapsack capacity that maximizes the value sum. This is called the **knapsack problem** or the **0,1 knapsack problem**.

We can model the 0,1 knapsack problem as an integer program. For each item $i \in [d]$, we introduce a binary variable x_i to indicate whetehr we choose item i or not. Specifically,

$$x_i = \begin{cases} 1, & \text{if item } i \text{ is taken in the knapsack} \\ 0, & \text{oterwise} \end{cases}.$$

Then the integer program is given by

$$\begin{aligned} \max \quad & \sum_{i=1}^d p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^d w_i x_i \leq B, \\ & x \in \{0,1\}^d \end{aligned}$$

Here, the constraint

$$\sum_{i=1}^{d} w_i x_i \leq B$$

is called the **knapsack constraint**. If we are allowed to choose multiple copies of an item, the problem is called the **unbounded knapsack problem**. For the unbounded case, variable x_i can take a nonnegative integer variable, i.e.

$$x_i \in \mathbb{Z}_+ \quad \text{is the number of copies of } i \text{ taken in the knapsack.}$$

In this case, the problem can be formulated as

$$\begin{aligned} \max \quad & \sum_{i=1}^{d} p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{d} w_i x_i \leq B, \\ & x \in \mathbb{Z}_+^d \end{aligned}$$

For the remainder of this section, we focus on the 0,1 knapsack problem where we take at most one copy of an item.

In fact, there is another formulation for the 0,1 knapsack problem. We refer the notion of **minimal covers**.

Definition 6.4. $C \subseteq [d]$ is a **minimal cover** if $\sum_{i \in C} w_i > B$ and $\sum_{i \in C \setminus \{j\}} w_i \leq B$ for all $j \in C$.

In words, C is a minimal cover if the weight sum of its items violates the knapsack capacity but all its proper subsets are under the capacity. Let us consider the following integer program.

$$\begin{aligned} \max \quad & \sum_{i=1}^{d} p_i x_i \\ \text{s.t.} \quad & \sum_{i \in C} x_i \leq |C| - 1, \quad \text{for every minimal cover } C, \\ & x \in \{0,1\}^d. \end{aligned} \tag{6.1}$$

Let K and K_C be defined as follows.

$$\begin{aligned} K &= \left\{ x \in \{0,1\}^d : \ \sum_{i=1}^{d} w_i x_i \leq B \right\}, \\ K_C &= \left\{ x \in \{0,1\}^d : \ \sum_{i \in C} x_i \leq |C| - 1, \quad \text{for every minimal cover } C \right\}. \end{aligned}$$

Proposition 6.5. *$K_C = K$, which means that (6.1) is a valid formulation of the 0,1 knapsack problem.*

Proof. If $\bar{x} \in K$, then $\bar{x}$ obviously belongs to K_C. Suppose that there exists $\bar{y} \in K_C$ such that $\bar{y} \notin K$. Then $\sum_{i=1}^{d} w_i \bar{y}_i > B$. Let $J = \{i : \ \bar{y}_i = 1\}$. Then $\sum_{i \in J} w_i > B$, so we can find a minimal cover C contained in J. Then $\sum_{i \in C} \bar{y}_i = |C| > |C| - 1$, which contradicts the assumption that $\bar{y} \in K_C$. □

Then the question is as to which formulation is better. To determine what is "better", we often compare the LP relaxations of two formulations. Basically, we determine which formulation has a tighter LP relaxation. Consider the following example.

Example 6.6. Consider the example where there are 3 items each of which has weight 3 while the knapsack capacity is 5. Then

$$\{x:\ 3x_1 + 3x_2 + 3x_3 \leq 5,\ 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3\}$$

corresponds to the LP relaxation of the first formulation. Next, note that $\{1, 2\}$, $\{2, 3\}$, and $\{3, 1\}$ are the minimal covers. Then

$$\{x:\ x_1 + x_2 \leq 1,\ x_2 + x_3 \leq 1,\ x_1 + x_3 \leq 1,\ 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3\}$$

is the relaxation of K_C. It is easy to show that the latter is strictly contained in the former, because $3x_1 + 3x_2 + 3x_3 \leq 9/2$ is valid for K_C. Therefore, the formulation based on minimal covers has a tighter relaxation.

The following example shows that the knapsack constraint based formulation can have a tighter LP relaxation than the minimal cover based formulation.

Example 6.7. Assume that each item has weight 1 and the knapsack capacity is 1. Then

$$\{x:\ x_1 + x_2 + x_3 \leq 1,\ 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3\}$$

corresponds to the LP relaxation of the knapsack constraint base formulation. On the other hand,

$$\{x:\ x_1 + x_2 \leq 1,\ x_2 + x_3 \leq 1,\ x_1 + x_3 \leq 1,\ 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3\}$$

corresponds to the other formulation. Note that the the former set is in fact the convex hull of the integer solutions $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and moreover, it is tighter than the latter set.