1 Outline

In this lecture, we study

- convex hull and reduction to linear programming,
- deriving a valid inequality for a two variable mixed integer linear set.

2 Convex hull and reduction to linear programming

A set $X \subseteq \mathbb{R}^d$ is **convex** if for any $u, v \in X$ and any $\lambda \in [0,1]$,

$$\lambda u + (1-\lambda)v \in X.$$

In words, the line segment joining any two points is entirely contained the set. In Figure 3.1, we have a convex set and a non-convex set.

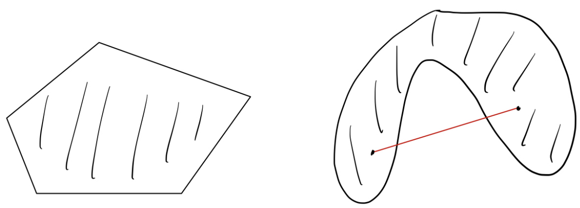

Figure 3.1: A convex set and a nonconvex set

Given $v^1, \ldots, v^k \in \mathbb{R}^d$, any linear combination $\lambda_1 v^1 + \cdots + \lambda_k v^k$ is a **convex combination** of $v^1, \ldots, v^k$ if

$$\sum_{i=1}^k \lambda_i = 1 \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for } i = 1, \ldots, k.$$

The convex combination of two distinct points u, v is the line segment $\{\lambda u + (1-\lambda)v : \ 0 \leq \lambda \leq 1\}$ connecting them.

The **convex hull** of a set $S \subseteq \mathbb{R}^d$, denoted $\mathrm{conv}(S)$, is the set of all convex combinations of points in S. By definition,

$$\mathrm{conv}(S) = \left\{ \sum_{i=1}^n \lambda_i v^i : \ \begin{array}{c} n \in \mathbb{N}, \ v^1, \ldots, v^n \in S, \\ \sum_{i=1}^n \lambda_i = 1, \ \lambda_1, \ldots, \lambda_n \geq 0 \end{array} \right\}.$$

$\mathrm{conv}(S)$ is always convex regardless of S. Figure 3.2 shows some examples of taking the convex hull of a (nonconvex) set.

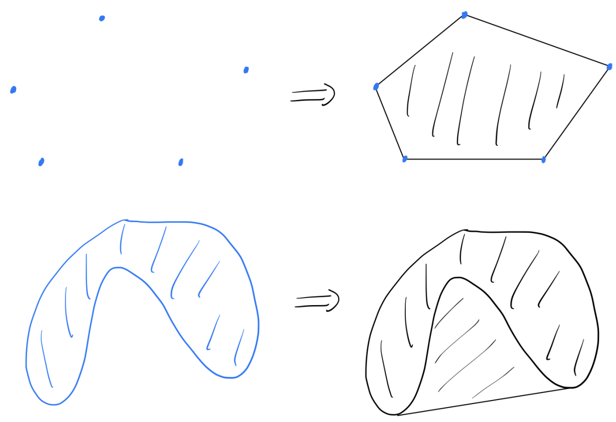

Figure 3.2: A convex set and a nonconvex set

For our integer program given by

$$\begin{aligned} \max \quad & c^\top x + h^\top y \\ \text{s.t.} \quad & Ax + Gy \le b, \\ & x \in \mathbb{Z}^d, \ y \in \mathbb{R}^p, \end{aligned} \tag{3.1}$$

we take the feasible region for the set S, whose convex hull is given by

$$\operatorname{conv}(S) = \operatorname{conv}\left(\left\{(x,y) \in \mathbb{Z}^d \times \mathbb{R}^p : \ Ax + Gy \le b\right\}\right).$$

Lemma 3.1. *The integer program (3.1) whose feasible region is given by $S \subseteq \mathbb{Z}^d \times \mathbb{R}^p$ satisfies*

$$\max\left\{c^\top x + h^\top y : \ (x,y) \in S\right\} = \max\left\{c^\top x + h^\top y : \ (x,y) \in \operatorname{conv}(S)\right\}.$$

Moreover, the supremum of $c^\top x + h^\top y$ is attained over S if and only if it is attained over $\operatorname{conv}(S)$.

Proof. Since $S \subseteq \operatorname{conv}(S)$, it is straightforward that

$$\max\left\{c^\top x + h^\top y : \ (x,y) \in S\right\} \le \max\left\{c^\top x + h^\top y : \ (x,y) \in \operatorname{conv}(S)\right\}.$$

Next we show that

$$\max\left\{c^\top x + h^\top y : \ (x,y) \in S\right\} \ge \max\left\{c^\top x + h^\top y : \ (x,y) \in \operatorname{conv}(S)\right\}$$

holds. Let $z^* = \max\left\{c^\top x + h^\top y : \ (x,y) \in S\right\}$. Then we may assume that z^* is finite. Let us consider

$$H = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^p : \ c^\top x + h^\top y \le z^*\right\}.$$

By definition, we have $S \subseteq H$. Moreover, as H is convex, it follows that $\operatorname{conv}(S) \subseteq H$. This implies that

$$\max\left\{c^\top x + h^\top y : \ (x,y) \in \operatorname{conv}(S)\right\} \le z^* = \max\left\{c^\top x + h^\top y : \ (x,y) \in S\right\},$$

which proves the desired inequality.

Assume that the supremum of $c^\top x + h^\top y$ is attaned at $(\bar{x}, \bar{y}) \in S$. Then

$$\max\left\{c^\top x + h^\top y:\ (x,y) \in S\right\} = c^\top \bar{x} + h^\top \bar{y}.$$

Note that $(\bar{x}, \bar{y}) \in \mathrm{conv}(S)$, and the first part implies that

$$\max\left\{c^\top x + h^\top y:\ (x,y) \in \mathrm{conv}(S)\right\} = c^\top \bar{x} + h^\top \bar{y}.$$

Now assume that the supremum of $c^\top x + h^\top y$ is attained at a point $(\bar{x}, \bar{y}) \in \mathrm{conv}(S)$. By the definition of $\mathrm{conv}(S)$, the point can be written as a convex combination of n points in S, given by $(x^1, y^1), \ldots, (x^n, y^n)$. As these n points also belong to $\mathrm{conv}(S)$, it follows that $c^\top x^i + h^\top y^i \leq c^\top \bar{x} + h^\top \bar{y}$ for all i. Moreover,

$$c^\top \bar{x} + h^\top \bar{y} = \sum_{i=1}^n \lambda_i (c^\top x^i + h^\top y^i)$$

for some $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i \in [d]} \lambda_i = 1$. Then

$$c^\top \bar{x} + h^\top \bar{y} = \sum_{i=1}^n \lambda_i (c^\top x^i + h^\top y^i) \leq \sum_{i=1}^n \lambda_i (c^\top \bar{x} + h^\top \bar{y}) = c^\top \bar{x} + h^\top \bar{y},$$

so the equalities hold throughout. Therefore, $c^\top x^i + h^\top y^i = c^\top \bar{x} + h^\top \bar{y}$ for all $i \in [n]$. □

By Lemma 3.1, solving the integer program (3.1) is equivalent to optimizing over the convex hull $\mathrm{conv}(S)$. By Meyer's theorem [Mey74] (we will discuss this later in this course), we know that there exists a system of rational linear inequalities $A'x + G'y \leq b'$ such that

$$\mathrm{conv}(S) = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^p:\ A'x + G'y \leq b'\right\}.$$

Consequently, the integer program $\max\left\{c^\top x + h^\top y:\ (x,y) \in S\right\}$ is equivalent to the linear program

$$\max\left\{c^\top x + h^\top y:\ A'x + G'y \leq b'\right\}$$

for some rational matrices A', G', b'. Therefore, we may say that integer programming reduces to linear programming. Wait, does this contradict our earlier discusstion that integer programming is NP-hard while linear programming is in class P? The answer is NO. The reason is that Meyer's theorem shows the *existence* of such a linear system, and in fact, computing the linear system that gives us the convex hull of S is in general hard.

3 Two-dimensional mixed integer linear set

Consider a mixed integer linear set given by

$$S = \{(x,y) \in \mathbb{Z} \times \mathbb{R}_+ :\ x - y \leq \beta\} \tag{3.2}$$

for some $\beta \in \mathbb{R}$. Note that

$$P = \{(x,y) \in \mathbb{R} \times \mathbb{R}_+ :\ x - y \leq \beta\}$$

corresponds to the LP relaxation of S defined by the two inequalities $y \geq 0$ and $x - y \leq \beta$. Figure 3.3 illustrates the mixed integer linear set S and its relaxation P. Let us characterzie the convex hull

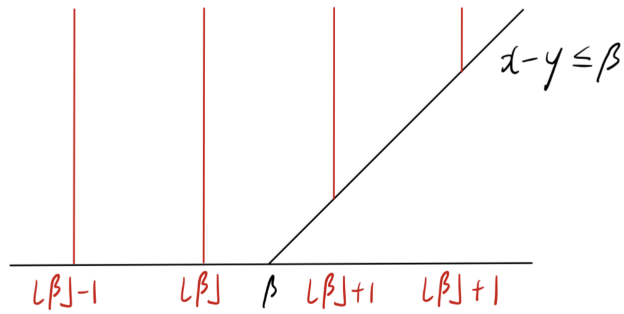

Figure 3.3: Illustration of S and P

of S.

Given a set $C \subseteq \mathbb{R}^d$, we say that $a^\top x \leq b$ where $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$ is **valid** and a **valid inequality** for C if

$$C \subseteq \left\{ x \in \mathbb{R}^d : \ a^\top x \leq b \right\}.$$

In words, inequality $a^\top x \leq b$ is valid for C if every point x in C satisfies the inequality. Here, the set

$$\left\{ x \in \mathbb{R}^d : \ a^\top x \leq b \right\}$$

is called a **half-space**, and the set

$$\left\{ x \in \mathbb{R}^d : \ a^\top x = b \right\}$$

is called a **hyperplane**.

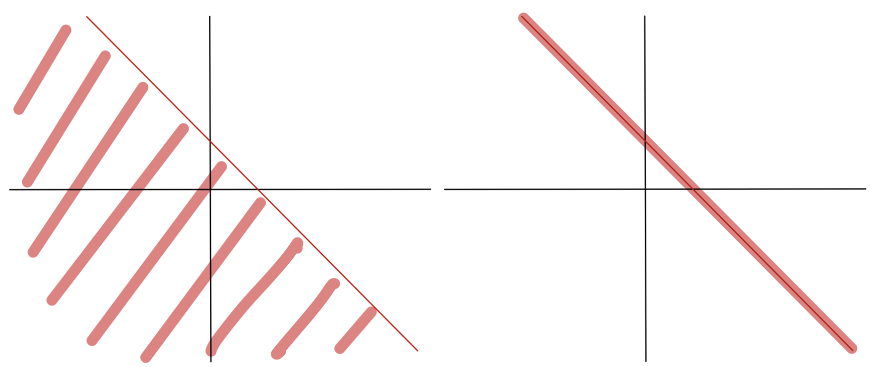

Figure 3.4: A half-space (left) and a hyperplane (right)

Lemma 3.2. *Let $f = \beta - \lfloor \beta \rfloor$ be the fractional part of β. Then the inequality*

$$x - \frac{1}{1-f} y \leq \lfloor \beta \rfloor \tag{3.3}$$

holds for any $(x, y) \in S$. In other words, the inequality is valid and a valid inequality for S.

Proof. Let $(x, y) \in S$. Then $x \leq \lfloor\beta\rfloor$ or $x \geq \lfloor\beta\rfloor + 1$. If $x \leq \lfloor\beta\rfloor$, then as $y \geq 0$, the inequality holds. If $x \geq \lfloor\beta\rfloor + 1$, then $x = \lfloor\beta\rfloor + k$ for some integer $k \geq 1$. Then $x - y \leq \beta$ implies that $y \geq k - f$, in which case

$$x - \frac{1}{1-f}y \leq \lfloor\beta\rfloor + k - \frac{k-f}{1-f} = \lfloor\beta\rfloor - \frac{(k-1)f}{1-f} \leq \lfloor\beta\rfloor,$$

as required. □

The inequality (3.3) in Lemma 3.2 holds at equality when

$$(x, y) = (\lfloor\beta\rfloor, 0),\ (\lfloor\beta\rfloor + 1, 1 - f).$$

Equivalently, the line defined by

$$x - \frac{1}{1-f}y = \lfloor\beta\rfloor$$

go through the two points $(\lfloor\beta\rfloor, 0)$ and $(\lfloor\beta\rfloor + 1, 1 - f)$, as shown in Figure 3.5.

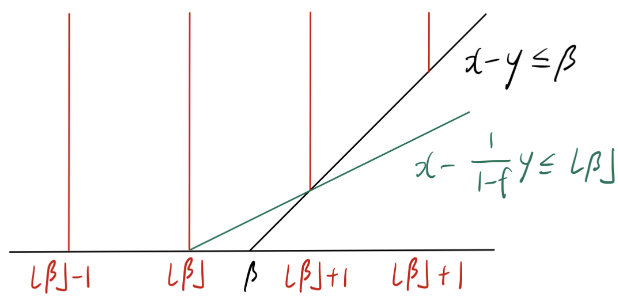

Figure 3.5: Valid inequality for S

We will see later that the **mixed integer rounding (MIR) cuts** by Nemhauser and Wolsey [NW90] are obtained based on the inequality (3.3) that is valid for the mixed integer linear set (3.2).

References

[CCZ14] Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. *Integer Programming*. Springer, 2014.

[Mey74] R. R. Meyer. On the existence of optimal solutions to integer and mixed integer programming problems, 1974. 2

[NW90] George L. Nemhauser and Laurence A. Wolsey. A recursive procedure to generate all cuts for 0-1 mixed integer programs. *Mathematical Programming*, 46:379–390, 1990. 3