1 Outline

In this lecture, we study

- branch-and-price method,
- Benders decomposition.

2 Branch-and-price based on Dantzig-Wolfe decomposition

Let us consider a mixed integer program

$$\begin{aligned} z_I \quad = \quad \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & Ex \leq f \\ & x \in \mathbb{Z}_+^d \times \mathbb{R}_+^p. \end{aligned} \tag{MIP}$$

Let Q be defined as

$$Q = \left\{ x \in \mathbb{Z}_+^d \times \mathbb{R}_+^p : \ Ax \leq b \right\}.$$

Suppose that $\mathrm{conv}(Q)$ can be expressed as

$$\mathrm{conv}(Q) = \mathrm{conv}\left\{v^1, \ldots, v^n\right\} + \mathrm{cone}\left\{r^1, \ldots, r^\ell\right\}$$

for some vectors $v^1, \ldots, v^n$ and $r^1, \ldots, r^\ell$. Recall that the Dantzig-Wolfe relaxation of (MIP) is given by

$$\begin{aligned} z_{\mathrm{LD}} \quad = \quad \max \quad & \sum_{k\in[n]} \left(c^\top v^k\right) \alpha_k + \sum_{h\in[\ell]} \left(c^\top r^h\right) \beta_k \\ \text{s.t.} \quad & \sum_{k\in[n]} \left(Ev^k\right) \alpha_k + \sum_{h\in[\ell]} \left(Er^h\right) \beta_k \leq f \\ & \sum_{k\in[n]} \alpha_k = 1 \\ & \alpha \in \mathbb{R}_+^k, \ \beta \in \mathbb{R}_+^\ell. \end{aligned} \tag{DW}$$

(DW) is a relaxation of (MIP), and moreover, we may impose the integrality constraints by adding

$$x_j = \sum_{k\in[n]} \alpha_k v_j^k + \sum_{h\in[\ell]} \beta_h r_j^h \in \mathbb{Z}, \quad j \in [d].$$

Consequently, we may apply the branch-and-bound framework to the Dantzig-Wolfe relaxation (DW). This approach is known as the **branch-and-price** method. The basic workflow is as follows.

1. Solve (DW) and obtain an optimal solution (α^*, β^*) which gives rise to

$$x^* = \sum_{k\in[n]} \alpha_k^* v_j^k + \sum_{h\in[\ell]} \beta_h^* r_j^h.$$

2. If $x_j^* \notin \mathbb{Z}$ for some $j \in [d]$, then we create two subproblems based on disjunction

$$\sum_{k\in[n]} \alpha_k v_j^k + \sum_{h\in[\ell]} \beta_h r_j^h \geq \lceil x_j^* \rceil \quad \text{or} \quad \sum_{k\in[n]} \alpha_k v_j^k + \sum_{h\in[\ell]} \beta_h r_j^h \leq \lfloor x_j^* \rfloor.$$

3. We repeat the above procedure for the subproblems.

3 Benders decomposition

We use the Lagrangian relaxation framework to deal with complicating constraints. In this section, we learn the **Benders reformulation** technique that can deal with complicating variables. Let us consider the following mixed-integer program.

$$\begin{aligned} z_I \quad = \quad \max \quad & c^\top x + q^\top y \\ \text{s.t.} \quad & Ax + Gy \leq b \\ & x \in \mathbb{Z}_+^d, \; y \in \mathbb{R}_+^p. \end{aligned} \tag{MIP}$$

Here, the integer variables x are complicating variables. If we fix the x part, then the optimization problem becomes

$$\begin{aligned} z_{LP}(x) \quad = \quad \max \quad & q^\top y \\ \text{s.t.} \quad & Gy \leq b - Ax \\ & y \in \mathbb{R}_+^p. \end{aligned}$$

Taking the dual of it, we deduce

$$\begin{aligned} \min \quad & u^\top (b - Ax) \\ \text{s.t.} \quad & G^\top u \geq q \\ & u \geq 0. \end{aligned}$$

Here, the feasible set of the dual does not depend on x. Let Q denote the feasible set of the dual:

$$Q = \left\{ u : \; G^\top u \geq q, \; u \geq 0 \right\}.$$

Suppose that Q can be expressed as

$$Q = \text{conv}\left\{v^1, \ldots, v^n\right\} + \text{cone}\left\{r^1, \ldots, r^\ell\right\}$$

for some vectors $v^1, \ldots, v^n$ and $r^1, \ldots, r^\ell$. We will prove the following theorem.

Theorem 23.1. (MIP) *can be reformulated as*

$$\begin{aligned} z_I \quad = \quad \max \quad & \eta \\ \textit{s.t.} \quad & \eta \leq c^\top x + (b - Ax)^\top v^k, \quad k \in [n] \\ & (b - Ax)^\top r^h \geq 0, \quad h \in [\ell] \\ & x \in \mathbb{Z}_+^d, \; \eta \in \mathbb{R}. \end{aligned} \tag{Benders}$$

To prove Theorem 23.1, we need the following projection theorem of Egon Balas.

Theorem 23.2. *Let $P = \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^p : Ax + Gy \leq b, y \geq 0\right\}$. Suppose that we can express $C = \left\{u:\ G^\top u \geq 0,\ u \geq 0\right\}$ as $C = \mathrm{cone}\left\{r^1, \ldots, r^\ell\right\}$ for some vectors $r^1, \ldots, r^\ell$. Then $\mathrm{proj}_x(P)$, the projection of P onto the x-space, is given by*

$$\mathrm{proj}_x(P) = \left\{x \in \mathbb{R}^d :\ (b - Ax)^\top r^h \geq 0,\ h \in [\ell]\right\}.$$

Proof. Let $\bar{x} \in \mathbb{R}^d$. Note that $\bar{x} \notin \mathrm{proj}_x(P)$ holds if and only if there is no $y \in \mathbb{R}^p$ that satisfies $Gy \leq b - A\bar{x}$ and $y \geq 0$. By Farkas' Lemma, the system $Gy \leq b - A\bar{x}, y \geq 0$ is infeasible if and only if there exists $u \in C$ such that $u^\top(b - A\bar{x}) < 0$. Since $C = \mathrm{cone}\left\{r^1, \ldots, r^\ell\right\}$, such a vector u exists if and only if $(b - A\bar{x})^\top r^h$ for some $h \in [\ell]$, in which case, $\bar{x} \notin \left\{x \in \mathbb{R}^d :\ (b - Ax)^\top r^h \geq 0,\ h \in [\ell]\right\}$. □

Let us prove Theorem 23.2.

Proof of Theorem 23.1. Let $P = \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^p : Ax + Gy \leq b, y \geq 0\right\}$. Note that

$$\begin{aligned} z_I \quad = \quad \max \quad & c^\top x + z_{LP}(x) \\ \text{s.t.} \quad & x \in \mathbb{Z}^d_+. \end{aligned}$$

Here, $z_{LP}(x) > -\infty$ if and only if there exists some $y \geq 0$ such that $Gy \leq b - Ax$, which is equivalent to $x \in \mathrm{proj}_x(P)$. Therefore, it follows that

$$\begin{aligned} z_I \quad = \quad \max \quad & c^\top x + z_{LP}(x) \\ \text{s.t.} \quad & x \in \mathrm{proj}_x(P) \cap \mathbb{Z}^d_+. \end{aligned}$$

Recall that $Q = \left\{u:\ G^\top u \geq q,\ u \geq 0\right\}$ and $Q = \mathrm{conv}\left\{v^1, \ldots, v^n\right\} + \mathrm{cone}\left\{r^1, \ldots, r^\ell\right\}$. Then $C = \left\{u:\ G^\top u \geq 0,\ u \geq 0\right\}$ is the recession cone of Q, so we have $C = \mathrm{cone}\left\{r^1, \ldots, r^\ell\right\}$. Then it follows from Theorem 23.2 that $\mathrm{proj}_x(P) = \left\{x \in \mathbb{R}^d :\ (b - Ax)^\top r^h \geq 0,\ h \in [\ell]\right\}$. Therefore, we deduce that

$$\begin{aligned} z_I \quad = \quad \max \quad & c^\top x + z_{LP}(x) \\ \text{s.t.} \quad & (b - Ax)^\top r^h \geq 0, \quad h \in [\ell] \\ & x \in \mathbb{Z}^d_+. \end{aligned}$$

Moreover, note that for any $x \in \mathrm{proj}_x(P)$, $z_{LP}(x) > -\infty$, so strong duality implies that

$$\begin{aligned} z_{LP}(x) \quad = \quad \min \quad & u^\top(b - Ax) \\ \text{s.t.} \quad & G^\top u \geq q \\ & u \geq 0. \end{aligned}$$

If $z_{LP}(x)$ is finite, then it means that Q is non-empty and

$$z_{LP}(x) = \min_{k \in [n]} \left\{(b - Ax)^\top v^k\right\}.$$

If $z_{LP}(x) = +\infty$, then Q is empty, so $z_{LP}(x) = \min_{k \in [n]} \left\{(b - Ax)^\top v^k\right\}$ also holds. Hence,

$$\begin{aligned} z_I \quad = \quad \max \quad & c^\top x + \min_{k \in [n]} \left\{(b - Ax)^\top v^k\right\} \\ \text{s.t.} \quad & (b - Ax)^\top r^h \geq 0, \quad h \in [\ell] \\ & x \in \mathbb{Z}^d_+. \end{aligned}$$

We may move the term $\min_{k\in[n]}\left\{(b-Ax)^\top v^k\right\}$ in the objective to constraints, after which we deduce that

$$\begin{aligned} z_I \quad = \quad \max \quad & \eta \\ \text{s.t.} \quad & \eta \leq c^\top x + \min_{k\in[n]}\left\{(b-Ax)^\top v^k\right\} \\ & (b-Ax)^\top r^h \geq 0, \quad h\in[\ell] \\ & x\in\mathbb{Z}_+^d,\ \eta\in\mathbb{R} \end{aligned}$$

which is equivalent to (Benders) as required. □

(Benders) is the **Benders reformulation** of (MIP). In general, the Benders reformulation has an enormous number of constraints. A natural approach is to work with a small subset of the constraints and add new ones as cutting planes. The **Benders decomposition algorithm**, which is nothing but the row generation framework for (Benders), is as follows.

At iteration t, we have $N_t\subseteq[n]$ and $L_t\subseteq[n]$. Then we solve

$$\begin{aligned} z_I^t \quad = \quad \max \quad & \eta \\ \text{s.t.} \quad & \eta \leq c^\top x + (b-Ax)^\top v^k, \quad k\in N_t \\ & (b-Ax)^\top r^h \geq 0, \quad h\in L_t \\ & x\in\mathbb{Z}_+^d,\ \eta\in\mathbb{R}. \end{aligned}$$

This is the **master problem** for the Benders decomposition algorithm. Assume that we get a solution (x^t,η^t) after solving the master problem at iteration t. Then the row generation framework attempts to find a violated inequality among

$$\begin{aligned} \eta \leq c^\top x + (b-Ax)^\top v^k, &\quad k\in[n]\setminus N_t, \\ (b-Ax)^\top r^h \geq 0, &\quad h\in[\ell]\setminus L_t. \end{aligned}$$

Hence, the question is

- does there exists $k_t\in[n]$ such that
$$\eta^t > c^\top x^t + (b-Ax^t)^\top v^{k_t}?$$
- does there exists $h_t\in[\ell]$ such that
$$(b-Ax^t)^\top r^{h_t} < 0?$$

To answer this, we solve

$$\begin{aligned} z_{LP}(x^t) \quad = \quad \max \quad & q^\top y \\ \text{s.t.} \quad & Gy \leq b - Ax^t \\ & y\in\mathbb{R}_+^p. \end{aligned}$$

This is the **subproblem** for the Benders decomposition algorithm. Note that if $z_{LP}(x^t)=+\infty$, then it means that for any $M>0$, there exists $y\geq 0$ such that $Ax^t+Gy\leq b$ and $c^\top x^t + q^\top y > M$, in which case $z_I=+\infty$. If $z_{LP}(x^t)$ is finite, then

$$z_{LP}(x^t) = \min_{k\in[n]}(b-Ax^t)^\top v^k = (b-Ax^t)^\top v^{k_t}$$

for some k_t. Hence, we deduce that

$$c^\top x^t + z_{LP}(x^t) = c^\top x^t + (b - Ax^t)^\top v^{k_t}.$$

Moreover, if $z_{LP}(x^t) = -\infty$, then the subproblem is infeasible, in which case, there exists $h_t \in [\ell]$ such that

$$(b - Ax^t)^\top r^{h_t} < 0.$$

Based on this discussion, we summarize the Benders decomposition algorithm as the following pseudo-code.

Algorithm 1 Benders decomposition algorithm

```
Initialize N_1 ⊆ [n] and L_1 ⊆ [ℓ].
for t = 1, ..., T do
    Solve the master problem with N_t ⊆ [n] and L_t ⊆ [ℓ].
    if z_I^t = −∞ then
        Conclude that (MIP) is infeasible.
    end if
    Let (x^t, η^t) be an optimal solution to the master problem.
    Solve the subproblem with x^t.
    if z_LP(x^t) = +∞ then
        Conclude that (MIP) is unbounded.
    else if z_LP(x^t) is finite then
        Let y^t be an optimal solution.
        Let k_t ∈ argmin_{k∈[n]} {(b − Ax^t)^⊤ v^k}.
        if c^⊤x^t + q^⊤y^t ≥ η^t then
            Conclude that (x^t, y^t) is an optimal solution to (MIP).
        else if c^⊤x^t + q^⊤y^t < η^t then
            Add constraint η ≤ c^⊤x + (b − Ax)^⊤ v^{k_t} (optimality cut).
            Update N_{t+1} = N_t ∪ {k_t}.
        end if
    else if z_LP(x^t) = −∞ then
        Then x^t ∉ proj_x(P) and there exists h_t ∈ [ℓ] such that (b − Ax)^⊤ r^{h_t} < 0.
        Add constraint (b − Ax)^⊤ r^{h_t} ≥ 0 (feasibility cut).
        Update L_{t+1} = L_t ∪ {h_t}.
    end if
end for
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