1 Outline

In this lecture, we study

- Dantzig-Wolfe decomposition based on the Lagrangian dual,
- Dantzig-Wolfe decomposition for binary programs,
- Dantzig-Wolfe decomposition for models with block diagonal structure,
- Column generation for the Dantzig-Wolfe reformulation.

2 Dantzig-Wolfe decomposition

Let us consider a mixed integer program

$$\begin{aligned} z_I \quad = \quad \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b \\ & Ex \le f \\ & x \in \mathbb{Z}_+^d \times \mathbb{R}_+^p. \end{aligned} \tag{MIP}$$

We will learn the **Dantzig-Wolfe decomposition** framework for solving the mixed-integer program.

2.1 Dantzig-Wolfe decomposition based on the Lagrangian dual

Let Q be defined as

$$Q = \left\{ x \in \mathbb{Z}_+^d \times \mathbb{R}_+^p : \ Ax \le b \right\}.$$

Assume that Q is nonempty and that A, b have rational entries. Let m be the number of rows of E, and take $\lambda \in \mathbb{R}_+^m$. Remember that we define the **Lagrangian relaxation** of (MIP) with respect to λ as follows.

$$\begin{aligned} z_{\mathrm{LR}}(\lambda) \quad = \quad \max \quad & c^\top x + \lambda^\top (f - Ex) \\ \text{s.t.} \quad & Ax \le b \\ & x \in \mathbb{Z}_+^d \times \mathbb{R}_+^p. \end{aligned} \tag{LR}$$

Moreover, recall that the **Lagrangian dual** of the mixed integer program (MIP) is defined as

$$z_{\mathrm{LD}} \quad = \quad \min \left\{ z_{\mathrm{LR}}(\lambda) : \ \lambda \ge 0 \right\}. \tag{LD}$$

We learned that (MIP) and (LD) are related according to the following characterization of (LD).

$$z_{\mathrm{LD}} = \max \left\{ c^\top x : \ Ex \le f, \ x \in \mathrm{conv}(Q) \right\}.$$

Furthermore, by the Minkowski-Weyl theorem, $\operatorname{conv}(Q)$ can be expressed as

$$\operatorname{conv}(Q) = \operatorname{conv}\left\{v^1, \ldots, v^n\right\} + \operatorname{cone}\left\{r^1, \ldots, r^\ell\right\}$$

where $v^1, \ldots, v^n$ are the extreme points of $\operatorname{conv}(Q)$ and $r^1, \ldots, r^\ell$ are the extreme rays of $\operatorname{conv}(Q)$. Then any point x in $\operatorname{conv}(Q)$ can be written as

$$x = \sum_{k\in[n]} \alpha_k v^k + \sum_{h\in[\ell]} \beta_h r^h$$

for some $\alpha \in \mathbb{R}^k_+$ and $\beta \in \mathbb{R}^\ell_+$ such that

$$\sum_{k\in[n]} \alpha_k = 1.$$

Based on this, it follows that

$$\begin{aligned} z_{\mathrm{LD}} \quad = \quad \max \quad & \sum_{k\in[n]} \left(c^\top v^k\right)\alpha_k + \sum_{h\in[\ell]} \left(c^\top r^h\right)\beta_k \\ \text{s.t.} \quad & \sum_{k\in[n]} \left(Ev^k\right)\alpha_k + \sum_{h\in[\ell]} \left(Er^h\right)\beta_k \leq f \\ & \sum_{k\in[n]} \alpha_k = 1 \\ & \alpha \in \mathbb{R}^k_+,\ \beta \in \mathbb{R}^\ell_+. \end{aligned} \tag{DW1}$$

Remember that the Lagrangian dual (LD) is a relaxation of (MIP). Hence, we refer to (DW1) as the **Dantzig-Wolfe relaxation** of (MIP). Moreover, we have

$$z_I = \max\left\{c^\top x :\ Ex \leq f,\ x \in \operatorname{conv}(Q),\ x_j \in \mathbb{Z}\ \forall j \in [d]\right\}.$$

Therefore, we deduce

$$\begin{aligned} z_I \quad = \quad \max \quad & \sum_{k\in[n]} \left(c^\top v^k\right)\alpha_k + \sum_{h\in[\ell]} \left(c^\top r^h\right)\beta_k \\ \text{s.t.} \quad & \sum_{k\in[n]} \left(Ev^k\right)\alpha_k + \sum_{h\in[\ell]} \left(Er^h\right)\beta_k \leq f \\ & \sum_{k\in[n]} \alpha_k = 1 \\ & \alpha \in \mathbb{R}^k_+,\ \beta \in \mathbb{R}^\ell_+ \\ & \sum_{k\in[n]} \alpha_k v^k_j + \sum_{h\in[\ell]} \beta_h r^h_j \in \mathbb{Z}, \quad j \in [d]. \end{aligned} \tag{DW2}$$

Here, the formulation (DW2) is referred to as the **Dantzig-Wolfe reformulation** of (MIP).

2.2 Dantzig-Wolfe decomposition as the dual of the Lagrangian dual

Recall that the Dantzig-Wolfe decomposition is given by

$$\begin{aligned}
\max \quad & \sum_{k\in[n]} \left(c^\top v^k\right)\alpha_k + \sum_{h\in[\ell]} \left(c^\top r^h\right)\beta_k \\
\text{s.t.} \quad & \sum_{k\in[n]} \left(Ev^k\right)\alpha_k + \sum_{h\in[\ell]} \left(Er^h\right)\beta_k \leq f \\
& \sum_{k\in[n]} \alpha_k = 1 \\
& \alpha \in \mathbb{R}^k_+,\ \beta \in \mathbb{R}^\ell_+.
\end{aligned}$$

is the equivalent representation of the Lagrangian dual. Let us take its dual. We use dual variable λ for the inequality constraint and dual variable μ for the equality constraint. Then we deduce

$$\begin{aligned}
\min \quad & \lambda^\top f + \mu \\
\text{s.t.} \quad & \mu + (Ev^k)^\top \lambda \geq c^\top v^k, \quad k \in [n] \\
& (Er^h)^\top \lambda \geq c^\top r^h, \quad h \in [\ell] \\
& \lambda \geq 0
\end{aligned}$$

Note that this is equivalent to

$$\begin{aligned}
\min \quad & \lambda^\top f + \mu \\
\text{s.t.} \quad & \mu \geq \max_{k\in[n]} \left\{ \left(c - E^\top \lambda\right)^\top v^k \right\} \\
& \lambda \in \mathrm{dom}(z_{\mathrm{LR}})
\end{aligned}$$

because

$$\mathrm{dom}(z_{\mathrm{LR}}) = \left\{ \lambda :\ \left(c - E^\top \lambda\right)^\top r^h \leq 0\ \forall h \in [\ell],\ \lambda \geq 0 \right\}.$$

Eliminating the variable μ, we obtain

$$\begin{aligned}
\min \quad & \lambda^\top f + \max_{k\in[n]} \left\{ \left(c - E^\top \lambda\right)^\top v^k \right\} \\
\text{s.t.} \quad & \lambda \in \mathrm{dom}(z_{\mathrm{LR}}).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
& \min_{\lambda\in\mathrm{dom}(z_{\mathrm{LR}})} \max_{k\in[n]} \left\{ \lambda^\top f + \left(c - E^\top \lambda\right)^\top v^k \right\} \\
&= \min_{\lambda\in\mathrm{dom}(z_{\mathrm{LR}})} \underbrace{\max_{k\in[n]} \left\{ c^\top v^k + \lambda^\top (f - Ev^k) \right\}}_{z_{\mathrm{LR}}(\lambda)} \\
&= \min \left\{ z_{\mathrm{LR}}(\lambda) :\ \lambda \in \mathrm{dom}(z_{\mathrm{LR}}) \right\} \\
&= z_{\mathrm{LD}}.
\end{aligned}$$

2.3 Dantzig-Wolfe decomposition for pure binary programs

Let us consider a pure binary integer program as follows.

$$\begin{aligned} z_I \quad = \quad \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & Ex \leq f \\ & x \in \{0,1\}^d. \end{aligned} \tag{BP}$$

We define Q as

$$Q = \left\{ x \in \{0,1\}^d : \; Ax \leq b \right\}.$$

Since Q is bounded and finite,

$$Q = \left\{ v^1, \ldots, v^n \right\}.$$

Then any point x in Q can be expressed as

$$x = \sum_{k \in [n]} \alpha_k v^k, \quad \sum_{k \in [n]} \alpha_k = 1, \quad \alpha \in \{0,1\}^n.$$

Then it follows that

$$\begin{aligned} z_I \quad = \quad \max \quad & \sum_{k \in [n]} \left(c^\top v^k \right) \alpha_k \\ \text{s.t.} \quad & \sum_{k \in [n]} \left(E v^k \right) \alpha_k \leq f \\ & \sum_{k \in [n]} \alpha_k = 1 \\ & \alpha \in \{0,1\}^n. \end{aligned}$$

This formulation is the Dantzig-Wolfe reformulation of (BP). Then the Dantzig-Wolfe relaxation of (BP) is

$$\begin{aligned} \max \quad & \sum_{k \in [n]} \left(c^\top v^k \right) \alpha_k \\ \text{s.t.} \quad & \sum_{k \in [n]} \left(E v^k \right) \alpha_k \leq f \\ & \sum_{k \in [n]} \alpha_k = 1 \\ & \alpha \geq 0 \end{aligned}$$

2.4 Problems with block diagonal structure

We consider the following optimization model

$$
\begin{array}{rlllll}
\max & c^{1\top}x^1 & + \; c^{2\top}x^2 + & \cdots & + \; c^{p\top}x^p & \\
\text{s.t.} & A^1x^1 & & & & \leq b^1 \\
 & & A^2x^2 & & & \leq b^2 \\
 & & & \ddots & & \\
 & & & & A^px^p & \leq b^p \\
 & E^1x^1 & + \; E^2x^2 + & \cdots & + \; E^px^d & \leq f \\
 & & x^j \in \{0,1\}^{d_j}, & & j \in [p]. &
\end{array}
\tag{22.1}
$$

For $j \in [p]$, let Q_j be defined as

$$
Q_j = \left\{ x^j \in \{0,1\}^{d_j} : \; A^jx^j \leq b^j \right\}.
$$

Here, Q_j is bounded and finite, so any point x^j in Q_j can be written as

$$
x^j = \sum_{v \in Q_j} \alpha_v^j v, \quad \sum_{v \in Q_j} \alpha_v^j = 1, \quad \alpha^j \in \{0,1\}^{|Q_j|}.
$$

Therefore, the Dantzig-Wolfe reformulation of (22.1) is given by

$$
\begin{aligned}
\max \quad & \sum_{v \in Q_1} \left(c^{1\top}v\right) \alpha_v^1 + \sum_{v \in Q_2} \left(c^{2\top}v\right) \alpha_v^2 + \cdots + \sum_{v \in Q_p} \left(c^{p\top}v\right) \alpha_v^p \\
\text{s.t.} \quad & \sum_{v \in Q_1} \left(E^1v\right) \alpha_v^1 + \sum_{v \in Q_2} \left(E^2v\right) \alpha_v^2 + \cdots + \sum_{v \in Q_p} \left(E^pv\right) \alpha_v^p \leq f \\
& \sum_{v \in Q_j} \alpha_v^j = 1, \quad j \in [p] \\
& \alpha^j \in \{0,1\}^{|Q_j|}, \quad j \in [p].
\end{aligned}
$$

Then the Dantzig-Wolfe relaxation of (22.1) is given by

$$
\begin{aligned}
\max \quad & \sum_{v \in Q_1} \left(c^{1\top}v\right) \alpha_v^1 + \sum_{v \in Q_2} \left(c^{2\top}v\right) \alpha_v^2 + \cdots + \sum_{v \in Q_p} \left(c^{p\top}v\right) \alpha_v^p \\
\text{s.t.} \quad & \sum_{v \in Q_1} \left(E^1v\right) \alpha_v^1 + \sum_{v \in Q_2} \left(E^2v\right) \alpha_v^2 + \cdots + \sum_{v \in Q_p} \left(E^pv\right) \alpha_v^p \leq f \\
& \sum_{v \in Q_j} \alpha_v^j = 1, \quad j \in [p] \\
& \alpha^j \geq 0, \quad j \in [p].
\end{aligned}
$$

Let us consider the special case where

- $c^1 = \cdots = c^p = c$,
- $E^1 = \cdots = E^p = E$,

- $Q^1 = \cdots = Q^p = Q$.

Then in the Dantzig-Wolfe relaxation, we may set

$$\alpha = \alpha^1 + \alpha^2 + \cdots + \alpha^p.$$

As a result, the Dantzig-Wolfe relaxation becomes

$$\begin{aligned} \max \quad & \sum_{v \in Q} \left(c^\top v\right) \alpha_v \\ \text{s.t.} \quad & \sum_{v \in Q} (Ev)\, \alpha_v \leq f \\ & \sum_{v \in Q} \alpha_v = p \\ & \alpha \geq 0. \end{aligned}$$

3 Column generation for solving the Dantzig-Wolfe reformulation

The Dantzig-Wolfe relaxation (DW1) has variables $\alpha_1, \ldots, \alpha_n$ for the extreme points of $\text{conv}(Q)$ and variables $\beta_1, \ldots, \beta_\ell$ for the extreme rays of $\text{conv}(Q)$. Therefore, n and ℓ are potentially very large. In this case, we may apply the column generation technique. Recall that the dual of (DW1) is given by

$$\begin{aligned} \min \quad & \lambda^\top f + \mu \\ \text{s.t.} \quad & \mu + (Ev^k)^\top \lambda \geq c^\top v^k, \quad k \in [n] \\ & (Er^h)^\top \lambda \geq c^\top r^h, \quad h \in [\ell] \\ & \lambda \geq 0. \end{aligned}$$

The column generation procedure works as follows. We start with $N \subseteq [n]$ and $L \subseteq [\ell]$. Then we have the master problem

$$\begin{aligned} \max \quad & \sum_{k \in N} \left(c^\top v^k\right) \alpha_k + \sum_{h \in L} \left(c^\top r^h\right) \beta_k \\ \text{s.t.} \quad & \sum_{k \in N} \left(Ev^k\right) \alpha_k + \sum_{h \in L} \left(Er^h\right) \beta_k \leq f \\ & \sum_{k \in N} \alpha_k = 1 \\ & \alpha \in \mathbb{R}^k_+, \ \beta \in \mathbb{R}^\ell_+. \end{aligned}$$

Given the corresponding dual solution (λ, μ), then the associated subproblem is given by

$$\max \left\{ \max_{k \in [n]} \left\{ (c - E^\top \lambda)^\top v^k - \mu \right\},\ \max_{h \in [\ell]} \left\{ (c - E^\top \lambda)^\top r^h \right\} \right\}.$$

If the value of the subproblem is strictly positive, then there exists $k \in [n] \setminus N$ or $h \in [\ell] \setminus L$ whose associated constraint in the dual is violated. Then we can add the corresponding variable. In fact, the subproblem can be equivalently solved by

$$\max \left\{ (c - E^\top \lambda)^\top x - \mu :\ x \in \text{conv}(Q) \right\} \quad \Leftrightarrow \quad \max \left\{ c^\top x + \lambda^\top (f - Ex) :\ x \in \text{conv}(Q) \right\}.$$

If this optimization problem is unbounded, then there must exist an extreme ray r^h for some $h \in [\ell] \setminus L$ such that $(Er^h)^\top \lambda < c^\top r^h$. If it has a strictly positive finite optimum, then there exists an extreme point v^k for some $k \in [n] \setminus N$ such that $\mu + (Ev^k)^\top \lambda < c^\top v^k$.