1 Outline

In this lecture, we study

- Lagrangian relaxation,
- Lagrangian dual,

2 Lagrangian relaxation

Let us consider a mixed integer program

$$\begin{aligned} z_I \quad = \quad \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b \\ & Ex \le f \\ & x \in \mathbb{Z}_+^d \times \mathbb{R}_+^p. \end{aligned} \tag{MIP}$$

Assume that $Ex \le f$ are **complicating constraints** in the sense that optimization without the constraints is easy. More precisely, assume that mixed integer programs of the following form are easy to solve:

$$\begin{aligned} \max \quad & h^\top x \\ \text{s.t.} \quad & Ax \le b \\ & x \in \mathbb{Z}_+^d \times \mathbb{R}_+^p. \end{aligned}$$

Let Q be defined as

$$Q = \left\{ x \in \mathbb{Z}_+^d \times \mathbb{R}_+^p : \ Ax \le b \right\}.$$

Assume that Q is nonempty and that A, b have rational entries. Let m be the number of rows of E, and take $\lambda \in \mathbb{R}_+^m$. Then we may define the **Lagrangian relaxation** of (MIP) with respect to λ as follows.

$$\begin{aligned} z_{\mathrm{LR}}(\lambda) \quad = \quad \max \quad & c^\top x + \lambda^\top (f - Ex) \\ \text{s.t.} \quad & Ax \le b \\ & x \in \mathbb{Z}_+^d \times \mathbb{R}_+^p. \end{aligned} \tag{LR}$$

Proposition 19.1. *$z_{LR}(\lambda) \ge z_I$ for any $\lambda \ge 0$.*

Proof. Let x^* be an optimal solution to (MIP). In particular, x^* satisfies

$$Ax^* \le b, \ Ex^* \le f, \ x^* \in \mathbb{Z}_+^d \times \mathbb{R}_+^p.$$

Then x^* is also feasible to the Lagrangian relaxation (LR). Moreover, as $Ex^* \le f$ and $\lambda \ge 0$, it follows that

$$\lambda^\top (f - Ex^*) \ge 0,$$

implying in turn that

$$z_{\mathrm{LR}}(\lambda) \geq c^\top x^* + \lambda^\top (f - Ex^*) \geq c^\top x^* = z_I.$$

Therefore, $z_{\mathrm{LR}}(\lambda) \geq z_I$ for any $\lambda \geq 0$, as required. □

What is the advantage of working with the Lagrangian relaxation? We assumed that $Ex \leq f$ are complicating constraints and that optimization over Q is easier than solving (MIP). Moreover, Proposition 19.1 implies that the Lagrangian relaxation (LR) provides a valid upper bound on (MIP).

Next, we define the **Lagrangian dual** of the mixed integer program (MIP).

$$z_{\mathrm{LD}} \quad = \quad \min\left\{z_{\mathrm{LR}}(\lambda): \ \lambda \geq 0\right\}. \tag{LD}$$

Hence, z_{LD} is the best possible/tightest upper bound on (MIP) achievable through Lagrangian relaxations.

Theorem 19.2. *z_{LD} satisfies the following.*

$$z_{LD} = \max\left\{c^\top x: \ Ex \leq f, \ x \in \mathrm{conv}(Q)\right\}.$$

Proof. As Q is a mixed-integer set defined by $Ax \leq b$, a rational system of linear inequalities, Meyer's theorem implies that

$$\mathrm{conv}(Q) = \left\{x \in \mathbb{R}^d \times \mathbb{R}^p: \ A'x \leq b'\right\}$$

for some A', b' with rational entries. First, observe that

$$\begin{aligned} z_{\mathrm{LR}}(\lambda) &= \max\left\{c^\top x + \lambda^\top (f - Ex): \ x \in Q\right\} \\ &= \max\left\{c^\top x + \lambda^\top (f - Ex): \ x \in \mathrm{conv}(Q)\right\} \\ &= \max\left\{c^\top x + \lambda^\top (f - Ex): \ A'x \leq b'\right\} \end{aligned}$$

where the second equality holds because $c^\top x + \lambda^\top (f - Ex)$ is a linear function. By strong LP duality,

$$\begin{aligned} z_{\mathrm{LR}}(\lambda) \quad = \quad \min \quad & b'^\top \mu + f^\top \lambda \\ \text{s.t.} \quad & A'^\top \mu \geq c - E^\top \lambda \\ & \mu \in \mathbb{R}_+^{m'} \end{aligned}$$

where m' is the number of rows of A'. This equality holds even when $z_{\mathrm{LR}}(\lambda)$ is unbounded. Then it follows that

$$\begin{aligned} z_{\mathrm{LD}} \quad = \quad \min \quad & b'^\top \mu + f^\top \lambda \\ \text{s.t.} \quad & A'^\top \mu + E^\top \lambda \geq c \\ & \mu \in \mathbb{R}_+^{m'}, \ \lambda \in \mathbb{R}_+^{m}. \end{aligned}$$

Then by strong LP duality again,

$$\begin{aligned} z_{\mathrm{LD}} \quad = \quad \max \quad & c^\top x \\ \text{s.t.} \quad & A'x \leq b' \\ & Ex \leq f \\ & x \in \mathbb{R}_+^d \times \mathbb{R}_+^p. \end{aligned}$$

Therefore,

$$z_{\mathrm{LD}} = \max\left\{c^\top x :\ Ex \le f,\ x \in \mathrm{conv}(Q)\right\},$$

as required. □

By the Minkowski-Weyl theorem, $\mathrm{conv}(Q)$ can be expressed as

$$\mathrm{conv}(Q) = \mathrm{conv}\left\{v^1, \ldots, v^n\right\} + \mathrm{cone}\left\{r^1, \ldots, r^\ell\right\}$$

where $v^1, \ldots, v^n$ are the extreme points of $\mathrm{conv}(Q)$ and $r^1, \ldots, r^\ell$ are the extreme rays of $\mathrm{conv}(Q)$. We may view $z_{\mathrm{LR}}(\lambda)$ as a function of λ.

Lemma 19.3. *The domain of z_{LR} is given by*

$$\mathrm{dom}(z_{LR}) = \left\{\lambda \in \mathbb{R}^m_+ :\ (c - E^\top \lambda)^\top r^j \le 0, \quad \forall j \in [\ell]\right\}.$$

Proof. Note that $z_{\mathrm{LR}}(\lambda)$ is finite if and only if $(c - E^\top\lambda)^\top r^j \le 0$ for all $j \in [\ell]$. □

Theorem 19.4. *z_{LR} is a convex piecewise linear function of λ over* $\mathrm{dom}(z_{LR})$.

Proof. Let $\lambda \in \mathrm{dom}(z_{\mathrm{LR}})$. Since

$$z_{\mathrm{LR}}(\lambda) = \max\left\{c^\top x + \lambda^\top(f - Ex) :\ x \in \mathrm{conv}(Q)\right\}$$

and

$$(c - E^\top\lambda)^\top r^j \le 0$$

for all $j \in [\ell]$, it follows that

$$z_{\mathrm{LR}}(\lambda) = \max\left\{f^\top\lambda + (c - E^\top\lambda)^\top v^j :\ j \in [n]\right\}.$$

Therefore, $z_{\mathrm{LR}}(\lambda)$ is the maximum of linear functions

$$c^\top v^j + (f - Ev^j)^\top\lambda, \quad j \in [n].$$

Hence, $z_{\mathrm{LR}}(\lambda)$ is convex piecewise linear. □

Theorem 19.5. *Let z_{LP} denote the optimal value of the LP relaxation of* (MIP)*. Then*

$$z_{IP} \le z_{LD} \le z_{LP}.$$

Proof. Note that

$$\begin{aligned}\left\{x \in \mathbb{R}^d \times \mathbb{R}^p :\ Ex \le f,\ x \in Q\right\} &\subseteq \left\{x \in \mathbb{R}^d \times \mathbb{R}^p :\ Ex \le f,\ x \in Q\right\} \\ &\subseteq \left\{x \in \mathbb{R}^d \times \mathbb{R}^p :\ Ex \le f,\ Ax \le b,\ x \ge 0\right\}.\end{aligned}$$

Therefore, $z_{IP} \le z_{LD} \le z_{LP}$, as required. □

Theorem 19.6. *We have $z_{LD} = z_{LP}$ if*

$$\mathrm{conv}(Q) = \left\{x \in \mathbb{R}^d_+ \times \mathbb{R}^p_+ :\ Ax \le b\right\}.$$