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1 Outline

In this lecture, we study

- Lift-and-project procedure,
- Lovász-Schrijver hierarchy,
- Sherali-Adams hierarchy.

2 Lift-and-project for mixed 0,1 programs

Let us consider

$$P = \left\{ x \in \mathbb{R}^d \times \mathbb{R}^p : \; A'x \leq b', \; 0 \leq x_j \leq 1 \; \forall j \in [d] \right\}.$$

Let us assume that $Ax \leq b$ consists of inequalities $A'x \leq b'$ and $0 \leq x_j \leq 1$ for $j \in [d]$. Hence, we have

$$P = \left\{ x \in \mathbb{R}^d \times \mathbb{R}^p : \; Ax \leq b \right\}.$$

We define S as

$$S = P \cap (\mathbb{Z}^d \times \mathbb{R}^p).$$

Since $x \in P$ satisfies $0 \leq x_j \leq 1$ for $j \in [d]$, it follows that

$$S \subseteq \{0,1\}^d \times \mathbb{R}^p.$$

Hence $x_1, \ldots, x_d$ are binary variables, and $x_{d+1}, \ldots, x_{d+p}$ are continuous variables.

For any $j \in [d]$, we may define a split as follows.

$$\{x \in P : \; x_j \leq 0\} \cup \{x \in P : \; x_j \geq 1\}.$$

Here, if $x \in P$ and $x_j \leq 0$, then we have $x_j = 0$ because any $x \in P$ satisfies $0 \leq x_j \leq 1$. Similarly, if $x \in P$ and $x_j \geq 1$, then we have $x_j = 1$. Therefore, the split is equivalent to

$$\{x \in P : \; x_j = 0\} \cup \{x \in P : \; x_j = 1\}.$$

Next we define P_j as the convex hull of the split:

$$P_j = \operatorname{conv}\left(\{x \in P : \; x_j = 0\} \cup \{x \in P : \; x_j = 1\}\right).$$

To obtain P_j, we may need to apply many split cuts. Instead, we attempt to deduce a description of P_j in a higher dimensional space than $\mathbb{R}^d \times \mathbb{R}^p$ by introducing some auxiliary variables. If this is possible, then P_j would be given by the projection of a polyhedron in a higher dimensional space. This idea of describing P_j in a higher dimensional space and taking projection is referred to as **lift-and-project** [BCC93].

The lift-and-project due to Balas, Ceria, and Cornuéjols [BCC93] procedure proceeds as follows.

1. Choose $j \in [d]$.
2. Generate the following nonlinear system from $0 \leq x_j \leq 1$ and $Ax \leq b$.

$$x_j(Ax - b) \leq 0,$$
$$(1 - x_j)(Ax - b) \leq 0.$$

   This is equivalent to

$$a_{ij}x_j^2 - b_i x_j + \sum_{k \neq j} a_{ik}x_k x_j \leq 0, \quad i \in [m],$$
$$\sum_{k=1}^{d+p} a_{ik}x_k - b_i - a_{ij}x_j^2 + b_i x_j - \sum_{k \neq j} a_{ik}x_k x_j \leq 0, \quad i \in [m],$$

   where $Ax \leq b$ consists of m constraints $\sum_{k=1}^{d+p} a_{ik}x_k \leq b_i$ for $i \in [m]$.
3. Substitute y_k for $x_k x_j$ for $k \neq j$ and x_j for x_j^2 and deduce

$$a_{ij}x_j - b_i x_j + \sum_{k \neq j} a_{ik}y_k \leq 0, \quad i \in [m],$$
$$\sum_{k=1}^{d+p} a_{ik}x_k - b_i - a_{ij}x_j + b_i x_j - \sum_{k \neq j} a_{ik}y_k \leq 0, \quad i \in [m].$$

   This system consists of linear inequalities only in terms of x and y variables. Let M_j denote the polyhedron that this system defines.
4. Project out the y variables from M_j to obtain $\mathrm{proj}_x(M_j)$.

Theorem 18.1 (Balas, Ceria, and Cornuéjols [BCC93])**.** *For every $j \in [d]$,*

$$\mathrm{proj}_x(M_j) = P_j = \mathrm{conv}\left(\{x \in P : \ x_j = 0\} \cup \{x \in P : \ x_j = 1\}\right).$$

Proof. By the theorem on union of polytopes due to Balas,

$$\mathrm{conv}\left(\{x \in P : \ x_j = 0\} \cup \{x \in P : \ x_j = 1\}\right)$$

can be described as the following linear system.

$$\begin{aligned} Ax^1 &\leq b\lambda \\ x_j^1 &= \lambda \\ Ax^0 &\leq b(1-\lambda) \\ x_j^0 &= 0 \\ x^1 + x^0 &= x \\ 0 \leq \lambda &\leq 1. \end{aligned}$$

In other words, we obtain the convex hull after projecting out the variables x^1, x^0, λ. Next we substitute $x^0 = x - x^1$ and replace x^1 by y. Then we deduce

$$\begin{aligned} Ay &\leq b\lambda \\ y_j &= \lambda \\ A(x-y) &\leq b(1-\lambda) \\ y_j &= x_j \\ 0 \leq \lambda &\leq 1. \end{aligned}$$

Substituting $\lambda = x_j$, it follows that

$$\begin{aligned} Ay &\le b x_j \\ A(x-y) &\le b(1-x_j) \\ y_j &= x_j \\ 0 \le x_j &\le 1. \end{aligned}$$

Here, $0 \le x_j \le 1$ is implied by $Ay \le bx_j$ and $A(x-y) \le b(1-x_j)$. Then we obtain

$$\begin{aligned} Ay - bx_j &\le 0 \\ (Ax-b) + (Ay - bx_j) &\le 0 \\ y_j &= x_j. \end{aligned}$$

This system is precisely what defines M_j. Therefore,

$$\operatorname{conv}\left(\{x \in P:\ x_j = 0\} \cup \{x \in P:\ x_j = 1\}\right) = \operatorname{proj}_x(M_j),$$

as required. □

Note that M_j is defined by $2m$ inequality constraints and $(d+p-1)+d = 2(d+p)-1$ variables because the number of y variables is $d+p-1$. Then we can consider

$$\bigcap_{j\in[d]} P_j = \bigcap_{j\in[d]} \operatorname{proj}_x(M_j).$$

By definition, we have

$$S \subseteq P^{\text{split}} \subseteq \bigcap_{j\in[d]} P_j \subseteq P$$

where P^{split} denotes the split closure of P. In particular, we obtain a tighter relaxation than P. Moreover, to obtain the intersection of P_j's, we need $2dm$ constraints and

$$d(d+p-1)+d+p$$

variables. Therefore, the encoding size of the intersection is polynomial in the encoding size of P.

The following theorem is referred to as the **sequential convexification theorem**.

Theorem 18.2 (Balas [Bal74, Bal98]). *The convex hull* $\operatorname{conv}(S)$ *of* S *can be obtained by sequentially taking the convex hull with respect to individual components, i.e.,*

$$\operatorname{conv}(S) = \left(\cdots((P_1)_2)_3\cdots\right)_d.$$

3 The Lovász-Schrijver construction

The lift-and-project procedure proposed by Lovász and Schrijver [LS91] is the following.

1. Generate the following nonlinear system from $0 \le x_j \le 1$ for $j = 1,\ldots,d$ and $Ax \le b$.

$$\begin{aligned} x_1(Ax-b) &\le 0, \\ (1-x_1)(Ax-b) &\le 0, \\ x_2(Ax-b) &\le 0, \\ (1-x_2)(Ax-b) &\le 0, \\ &\vdots \\ x_d(Ax-b) &\le 0, \\ (1-x_d)(Ax-b) &\le 0. \end{aligned}$$

2. Substitute $y_{kj} = x_k x_j$ for every pair of distinct $k, j \in [d+p]$, set $y_{ij} = y_{ji}$, and substitute $x_j = x_j^2$ for $j \in [d]$. Let $M(P)$ denote the resulting polyhedron in the (x, y)-space.

3. Project out the y variables, and obtain $N(P) = \mathrm{proj}_x(M(P))$. Here, $N(P)$ is a polyhedron.

Theorem 18.3 (Lovász and Schrijver [LS91])**.** *The Lovász-Schrijver construction satisfies*

$$N(P) \subseteq \bigcap_{j \in [d]} P_j.$$

We may further strengthen the Lovász-Schrijver construction. Let Y denote the $d \times d$ matrix whose off-diagonal entries are given by y_{ij} for $i \neq j \in [d]$ and whose diagonal entries are given by x_i for $i \in [d]$. Note that $y_{ij} = x_i x_j$ and $x_i = x_i^2$. This implies that

$$Y = xx^\top.$$

Then we know that Y is positive semidefinite (PSD). Based on this fact, we define

$$M^+(P) = \{(x, y) \in M(P) : \ Y \succeq 0\}.$$

Then we define $N^+(P)$ as

$$N^+(P) = \mathrm{proj}_x(M^+(P)).$$

Here, $M^+(P)$ is not a polyhedron anymore, due to the PSD constraint $Y \succeq 0$. Nevertheless, it follows from the definition that

$$N^+(P) \subseteq N(P).$$

4 The Sherali-Adams construction

The lift-and-project procedure proposed by Sherali and Adams [SA90] is the following.

1. Choose a number $t \in \{1, \ldots, d\}$.

2. For any pair of disjoint subsets J_1 and J_2 such that $J_1 \cup J_2 \subseteq [d]$ and $|J_1 \cup J_2| = t$, generate the following nonlinear system.

$$\prod_{j \in J_1} x_j \prod_{j \in J_2} (1 - x_j)(Ax - b) \leq 0.$$

3. Substitute x_j for x_j^2, substitute $w_J = \prod_{j \in J} x_j$ for any $J \subseteq [d]$, and substitute $v_{Jk} = x_k \prod_{j \in J} x_j$ for any $J \subseteq [d]$ and $k \geq d+1$. Let $X_t(P)$ denote the resulting polyhedron in the (x, w, v)-space.

4. Project out the w and v variables, and obtain $K_t(P)$. Here, $K_t(P)$ is a polyhedron.

Theorem 18.4 (Sherali and Adams [SA90])**.** $K_d(P) = \mathrm{conv}(S)$*.*

Theorem 18.5 (Balas, Ceria, and Cornuéjols [BCC93])**.** *For* $t = 1, \ldots, d$*,*

$$K_t(P) \subseteq \left(\cdots((P_1)_2)_3 \cdots\right)_t.$$

Theorem 18.6. *For* $t = 1, \ldots, d$*,*

$$K_t(P) \subseteq \underbrace{N\left(N\left(\cdots N(P)\cdots\right)\right)}_{t \text{ recursive applications}}$$

References

[Bal74] Egon Balas. Disjunctive programming: properties of the convex hull of feasible points. *GSIA Management Science Research Report MSSR 348, Carnegie Mellon University*, 1974. 18.2

[Bal98] E. Balas. Disjunctive programming: Properties of the convex hull of feasible solutions. *Discrete Applied Mathematics*, 89:3–44, 1998. 18.2

[BCC93] Egon Balas, Sebastián Ceria, and Gérard Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Mathematical Programming*, 58:295–324, 1993. 2, 18.1, 18.5

[LS91] L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0-1 optimization. *SIAM Journal on Optimization*, 1(2):166–190, 1991. 3, 18.3

[SA90] S. Sherali and W. Adams. A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. *SIAM Journal on Discrete Mathematics*, 3(3):411–430, 1990. 4, 18.4