1 Outline

In this lecture, we study

- split cuts,
- Gomory's mixed-integer cuts,
- relationships between Gomory's mixed-integer cuts, Gomory's fractional cuts, and split cuts.

2 Split cuts

Consider a mixed-integer program given as follows.

$$\begin{aligned} \min \quad & c^\top x + h^\top y \\ \text{s.t.} \quad & Ax + Gy \leq b \\ & x \in \mathbb{Z}^d, \ y \in \mathbb{R}^p \end{aligned}$$

where A, G, b have rational entries. Then

$$P = \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^p : \ Ax + Gy \leq b \right\}$$

is a rational polyhedron and the feasible set of the LP relaxation. Moreover,

$$S = P \cap (\mathbb{Z}^d \times \mathbb{R}^p)$$

is the set of solutions to the mixed-integer program.

Let $\pi \in \mathbb{Z}^d$ and $\pi_0 \in \mathbb{Z}$. Note that for any $x \in \mathbb{Z}^d$, we know that $\pi^\top x$ is an integer. Therefore, any $x \in \mathbb{Z}^d$ satisfies $\pi^\top x \leq \pi_0$ or $\pi^\top x \geq \pi_0 + 1$. Note that $\mathbb{Z}^d \times \mathbb{R}^p$ can be partitioned as

$$\mathbb{Z}^d \times \mathbb{R}^p = \left\{ (x, y) \in \mathbb{Z}^d \times \mathbb{R}^p : \ \pi^\top x \leq \pi_0 \right\} \cup \left\{ (x, y) \in \mathbb{Z}^d \times \mathbb{R}^p : \ \pi^\top x \geq \pi_0 + 1 \right\}.$$

Similarly, it follows that

$$S = \left\{ (x, y) \in S : \ \pi^\top x \leq \pi_0 \right\} \cup \left\{ (x, y) \in S : \ \pi^\top x \geq \pi_0 + 1 \right\}.$$

Motivated by this, we define two polyhedron as follows.

$$\begin{aligned} \Pi_1 &= \left\{ (x, y) \in P : \ \pi^\top x \leq \pi_0 \right\}, \\ \Pi_2 &= \left\{ (x, y) \in P : \ \pi^\top x \geq \pi_0 + 1 \right\}. \end{aligned}$$

Note that

$$S \subseteq \Pi_1 \cup \Pi_2.$$

Here, $\Pi_1 \cup \Pi_2$ is a subset of P as shown in Figure 17.1, and therefore, $\Pi_1 \cup \Pi_2$ is a stronger relaxation of S than P.

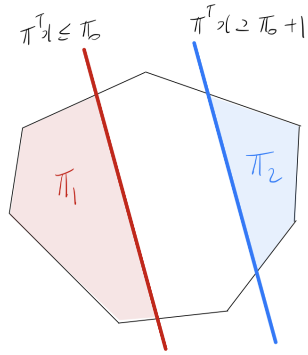

Figure 17.1: Splitting a polyhedron

We refer to an inequality $\alpha^\top x + \beta^\top y \le \gamma$ that is valid for $\Pi_1 \cup \Pi_2$ as a **split cut** for P. As shown in Figure 17.2, a split cut may cut off some part of the polyhedron P.

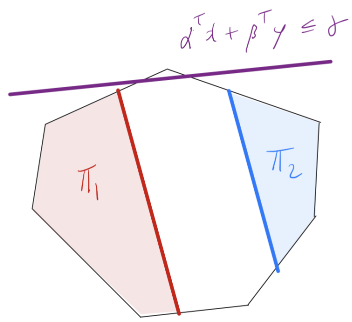

Figure 17.2: Split cut for polyhedron P

The set between two parallel hyperplanes, given by

$$\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^p : \ \pi_0 \le \pi^\top x \le \pi_0 + 1\}$$

is called a **split set**. We refer to $\Pi_1 \cup \Pi_2$ obtained from the polyhedron P and $(\pi, \pi_0) \in \mathbb{Z}^d \times \mathbb{Z}$ as a **split**. Note that splits are defined with the integeer variables, not with continuous variables.

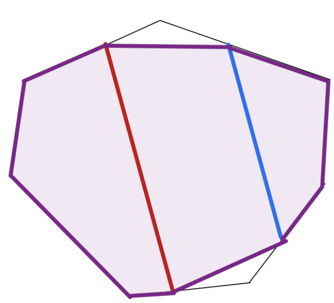

Figure 17.3: Applying all splits cuts for a split

If we apply all split cuts from the split with (π, π_0), then we obtain $\mathrm{conv}(\Pi_1 \cup \Pi_2)$ as in Figure 17.3.

Recall that an inequality $\pi^\top x \le \pi_0$ is a Chvátal-Gomory cut if

$$P \cap \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^p : \ \pi^\top x \ge \pi_0 + 1\right\} = \emptyset.$$

In fact, a Chvátal-Gomory cut is a split cut when one of Π_1 and Π_2 is empty (see Figure 17.4).

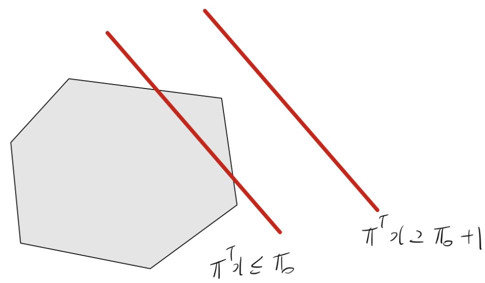

Figure 17.4: Chvátal-Gomory cut as a split cut

We may take all possible split cuts from every possible $(\pi, \pi_0) \in \mathbb{Z}^d \times \mathbb{Z}$. The resulting set is called the **split closure** of the polyhedron P. The split closure of P is given by

$$P^{(1)} = \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^d\times\mathbb{Z}} \operatorname{conv}\left(\Pi_1^{(\pi,\pi_0)} \cup \Pi_2^{(\pi,\pi_0)}\right)$$

where $\Pi_1^{(\pi,\pi_0)} \cup \Pi_2^{(\pi,\pi_0)}$ denotes the split associated with $(\pi, \pi_0) \in \mathbb{Z}^d \times \mathbb{Z}$.

Theorem 17.1 (Cook, Kannan, and Schrijver [CKS90])**.** *The split closure of any rational polyhedron is a rational polyhedron.*

Then the split closure $P^{(1)}$ is a rational polyhedron. Therefore, we may recursively apply the procedure of taking the split closure. Let $P^{(k)}$ denote the k**th split closure** of P, that is, the split closure of $P^{(k-1)}$.

Example 17.2. Consider a polyhedron

$$P = \left\{(x_1, x_2, y) \in \mathbb{R}^3_+ : \ x_1 \ge y, \ x_2 \ge y, \ x_1 + x_2 + 2y \le 2\right\}$$

and the associated mixed-integer set

$$S = P \cap (\mathbb{Z}^2 \times \mathbb{R}).$$

Note that P is a convex combination of 4 points as follows.

$$P = \operatorname{conv}\left(\left\{(0,0,0), (2,0,0), (0,2,0), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\}\right).$$

Here, P has an apex $(1/2, 1/2, 1/2)$. Moreover,

$$\begin{aligned} S &= \{(0,0,0), (2,0,0), (0,2,0)\} \\ &= \left\{(x_1, x_2, y) \in \mathbb{R}^3_+ : \ x_1 \ge y, \ x_2 \ge y, \ y \le 0\right\}. \end{aligned}$$

Therefore, to obtain the convex hull of S, we need to deduce inequality $y \le 0$. However, we can argue that the kth split closure of $P^{(k)}$ for any finite k contains a point of the form $(1/2, 1/2, t)$ for some $t > 0$.

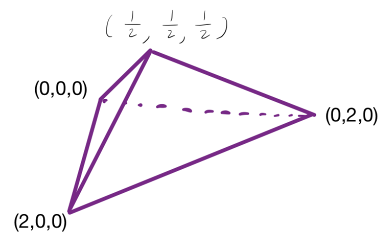

Figure 17.5: P as the convex hull of some points

3 Gomory's mixed-integer cuts

We consider the following mixed-integer set

$$S = \left\{ (x,y) \in \mathbb{Z}_+^d \times \mathbb{R}_+^p : \sum_{i\in[d]} a_i x_i + \sum_{j\in[p]} g_j y_j = b \right\} \subseteq \mathbb{Z}_+^d \times \mathbb{R}_+^p.$$

Let $f_0, f_1, \ldots, f_d$ be defined as

$$f_0 = b - \lfloor b \rfloor, \quad f_i = a_i - \lfloor a_i \rfloor \quad \text{for } i \in [d].$$

We assume that

$$0 < f_0 < 1,$$

i.e., b is not an integer, so that we may generate a nontrivial cut. Then

$$\sum_{i\in[d]} a_i x_i + \sum_{j\in[p]} g_j y_j = b$$

is equivalent to

$$\sum_{i\in[d]} f_i x_i + \sum_{j\in[p]} g_j y_j = f_0 + \left(\lfloor b \rfloor - \sum_{i\in[d]} \lfloor a_i \rfloor x_i \right).$$

This is further reduced to

$$\sum_{i\in[d]: f_i \le f_0} f_i x_i + \sum_{i\in[d]: f_i > f_0} (f_i - 1) x_i + \sum_{j\in[p]} g_j y_j = f_0 + \left(\lfloor b \rfloor - \sum_{i\in[d]: f_i \le f_0} \lfloor a_i \rfloor x_i - \sum_{i\in[d]: f_i > f_0} (\lfloor a_i \rfloor + 1) x_i \right).$$

Therefore, we deduce that

$$S \subseteq \left\{ (x,y) \in \mathbb{Z}_+^d \times \mathbb{R}_+^p : \sum_{i\in[d]: f_i \le f_0} f_i x_i + \sum_{i\in[d]: f_i > f_0} (f_i - 1) x_i + \sum_{j\in[p]} g_j y_j = f_0 + k \quad \text{for some integer } k \right\}.$$

The set on the right-hand side is contained in

$$\left\{ (x,y) \in \mathbb{Z}_+^d \times \mathbb{R}_+^p : \sum_{i\in[d]: f_i \le f_0} f_i x_i + \sum_{i\in[d]: f_i > f_0} (f_i - 1) x_i + \sum_{j\in[p]} g_j y_j \ge f_0 \right\}$$
$$\bigcup \left\{ (x,y) \in \mathbb{Z}_+^d \times \mathbb{R}_+^p : \sum_{i\in[d]: f_i \le f_0} f_i x_i + \sum_{i\in[d]: f_i > f_0} (f_i - 1) x_i + \sum_{j\in[p]} g_j y_j \le f_0 - 1 \right\},$$

which is equivalent to

$$\left\{(x,y)\in\mathbb{Z}_+^d\times\mathbb{R}_+^p:\ \sum_{i\in[d]:f_i\le f_0}\frac{f_i}{f_0}x_i+\sum_{i\in[d]:f_i>f_0}\frac{f_i-1}{f_0}x_i+\sum_{j\in[p]}\frac{g_j}{f_0}y_j\ge 1\right\}$$
$$\bigcup\left\{(x,y)\in\mathbb{Z}_+^d\times\mathbb{R}_+^p:\ \sum_{i\in[d]:f_i\le f_0}\frac{-f_i}{1-f_0}x_i+\sum_{i\in[d]:f_i>f_0}\frac{1-f_i}{1-f_0}x_i+\sum_{j\in[p]}\frac{-g_j}{1-f_0}y_j\ge 1\right\},$$

Then

$$\sum_{i\in[d]:f_i\le f_0}\max\left\{\frac{f_i}{f_0},\frac{-f_i}{1-f_0}\right\}x_i+\sum_{i\in[d]:f_i>f_0}\max\left\{\frac{f_i-1}{f_0},\frac{1-f_i}{1-f_0}\right\}x_i+\sum_{j\in[p]}\max\left\{\frac{g_j}{f_0},\frac{-g_j}{1-f_0}\right\}y_j\ge 1$$

is a valid inequality for S. Since $0\le f_i<1$ for $i\in[p]$. Here, the inequality is equal to

$$\sum_{i\in[d]:f_i\le f_0}\frac{f_i}{f_0}x_i+\sum_{i\in[d]:f_i>f_0}\frac{1-f_i}{1-f_0}x_i+\sum_{j\in[p]:g_j\ge 0}\frac{g_j}{f_0}y_j+\sum_{j\in[p]:g_j<0}\frac{-g_j}{1-f_0}y_j\ge 1.$$

This inequality is a **Gomory's mixed-integer (GMI) cut**.

3.1 Comparison with Gomory's fractional cuts

Recall that we considered sets of the form

$$S=\left\{x\in\mathbb{Z}_+^d:\ \sum_{i\in[d]}a_ix_i=b\right\}.$$

Recall that Gomory's fractional cut is given by

$$\sum_{i\in[d]}(a_i-\lfloor a_i\rfloor)\,x_i\ge b-\lfloor b\rfloor.$$

Then Gomory's fractional cut is equivalent to

$$\sum_{i\in[d]}\frac{f_i}{f_0}x_i\ge 1$$

where $f_i=a_i-\lfloor a_i\rfloor$ for $i\in[d]$ and $f_0=b-\lfloor b\rfloor$. On the other hand, Gomory's mixed-integer cut for the set S has the form

$$\sum_{i\in[d]:f_i\le f_0}\frac{f_i}{f_0}x_i+\sum_{i\in[d]:f_i>f_0}\frac{1-f_i}{1-f_0}x_i\ge 1.$$

In fact, if $f_i>f_0$, then

$$\frac{f_i}{f_0}>\frac{1-f_i}{1-f_0}.$$

This indicates that Gomory's mixed-integer cuts dominate Gomory's fractional cuts.

3.2 Connection to split cuts

Let us consider again the mixed-integer set given by

$$S = \left\{ (x, y) \in \mathbb{Z}_+^d \times \mathbb{R}_+^p : \sum_{i \in [d]} a_i x_i + \sum_{j \in [p]} g_j y_j = b \right\} \subseteq \mathbb{Z}_+^d \times \mathbb{R}_+^p.$$

Let $(\pi, \pi_0) \in \mathbb{Z}^d \times \mathbb{Z}$ be defined as follows.

$$\pi_i = \begin{cases} \lfloor a_i \rfloor, & \text{if } f_i \leq f_0 \\ \lfloor a_i \rfloor + 1, & \text{if } f_i > f_0 \end{cases}, \quad \pi_0 = \lfloor b \rfloor.$$

Then (π, π_0) defines split $\Pi_1 \cup \Pi_2$ given by

$$\Pi_1 = \left\{ (x, y) \in \mathbb{R}_+^d \times \mathbb{R}_+^p : \sum_{i \in [d]} a_i x_i + \sum_{j \in [p]} g_j y_j = b, \sum_{i \in [d]: f_i \leq f_0} \lfloor a_i \rfloor x_i + \sum_{i \in [d]: f_i > f_0} (\lfloor a_i \rfloor + 1) x_i \leq \lfloor b \rfloor \right\},$$

$$\Pi_2 = \left\{ (x, y) \in \mathbb{R}_+^d \times \mathbb{R}_+^p : \sum_{i \in [d]} a_i x_i + \sum_{j \in [p]} g_j y_j = b, \sum_{i \in [d]: f_i \leq f_0} \lfloor a_i \rfloor x_i + \sum_{i \in [d]: f_i > f_0} (\lfloor a_i \rfloor + 1) x_i \geq \lfloor b \rfloor + 1 \right\}$$

For Π_1, subtracting the inequality from the equality, we deduce that

$$\sum_{i \in [d]: f_i \leq f_0} f_i x_i + \sum_{i \in [d]: f_i > f_0} (f_i - 1) x_i + \sum_{j \in [p]} g_j y_j \geq f_0$$

is valid for Π_1. For Π_2, subtracting the inequality from the equality, we deduce that

$$\sum_{i \in [d]: f_i \leq f_0} f_i x_i + \sum_{i \in [d]: f_i > f_0} (f_i - 1) x_i + \sum_{j \in [p]} g_j y_j \leq f_0 - 1$$

is valid for Π_2. The inequalities are equivalent to

$$\begin{aligned} \sum_{i \in [d]: f_i \leq f_0} \frac{f_i}{f_0} x_i + \sum_{i \in [d]: f_i > f_0} \frac{f_i - 1}{f_0} x_i + \sum_{j \in [p]} \frac{g_j}{f_0} y_j \geq 1 \\ \sum_{i \in [d]: f_i \leq f_0} \frac{-f_i}{1 - f_0} x_i + \sum_{i \in [d]: f_i > f_0} \frac{1 - f_i}{1 - f_0} x_i + \sum_{j \in [p]} \frac{-g_j}{1 - f_0} y_j \geq 1 \end{aligned}$$

where the first inequality is valid for Π_1 and the second inequality is valid for Π_2. Then we can argue that

$$\sum_{i \in [d]: f_i \leq f_0} \max\left\{ \frac{f_i}{f_0}, \frac{-f_i}{1 - f_0} \right\} x_i + \sum_{i \in [d]: f_i > f_0} \max\left\{ \frac{f_i - 1}{f_0}, \frac{1 - f_i}{1 - f_0} \right\} x_i + \sum_{j \in [p]} \max\left\{ \frac{g_j}{f_0}, \frac{-g_j}{1 - f_0} \right\} y_j \geq 1$$

is valid for $\Pi_1 \cup \Pi_2$. In fact, the inequality is precisely Gomory's mixed-integer cut for the set S.

3.3 Gomory's mixed-integer closure

Let us consider a rational polyhedron given by

$$P = \left\{ (x, y) \in \mathbb{R}^d_+ \times \mathbb{R}^p_+ : \ Ax + Gy \leq b \right\}.$$

By adding nonnegative slack variables s, we deduce that

$$P = \left\{ (x, y) \in \mathbb{R}^d_+ \times \mathbb{R}^p_+ : \ \exists s \in \mathbb{R}^m_+ \text{ s.t. } Ax + Gy + s = b \right\}$$

where m is the number of inequalities in the system $Ax + Gy \leq b$. Then we may take any linear combination of the equality constraints in $Ax + Gy + s = b$ by taking a multiplier λ:

$$(A^\top \lambda)x + (G^\top \lambda)y + \lambda^\top s = b^\top \lambda.$$

Then we can generate Gomory's mixed-integer cut associated with this equation. Repeating this procedure for all every possible λ, we may apply all possible Gomory's mixed-integer cuts. We refer to the resulting set as the **mixed integer closure** of P.

Theorem 17.3. *The mixed integer closure of P coincides with the split closure of P.*

References

[CKS90] W.J. Cook, R. Kannan, and A. Schrijver. Chvátal closures for mixed integer programming problems. *Mathematical Programming*, 47:155–174, 1990. 17.1