1 Outline

In this lecture, we study

- Chvátal closure,
- Gomory's fractional cuts,

2 Chvátal-Gomory cuts

2.1 Geometric view

Let $P = \{x \in \mathbb{R}^d : \ Ax \leq b\}$ be a polyhedron and $S = P \cap \mathbb{Z}^d$ be the set of integer solutions in P. Let $c \in \mathbb{Z}^d$. Then

$$c^\top x \leq \max\left\{c^\top y : \ y \in P\right\}$$

is valid for P. Let $d \in \mathbb{Z}$ be defined as

$$d = \lfloor \max\left\{c^\top y : \ y \in P\right\} \rfloor.$$

Then it follows that

$$P \cap \{x \in \mathbb{R}^d : \ c^\top x \geq d+1\} = \emptyset.$$

Moreover, $c^\top x \leq d$ is valid for $\text{conv}(S)$.

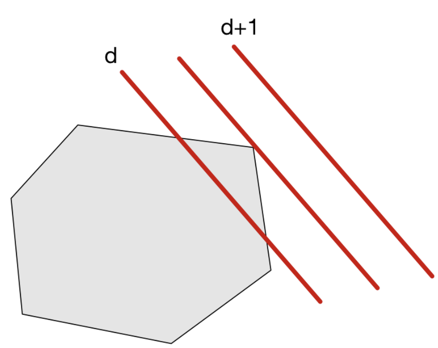

Figure 16.1: Geometric view of Chvátal-Gomory cuts

Proposition 16.1. *$c^\top x \leq d$ is a Chvátal-Gomory cut, and every Chvátal-Gomory cut can be obtained this way.*

Proof. Let $\beta = \max\{c^\top x : x \in P\}$. Then $\beta < d+1$. Moreover, $c^\top x \leq \beta$ is valid for P. Then by Strong duality, there exists $\lambda \in \mathbb{R}^m_+$ such that $A^\top \lambda = c$ and $b^\top \lambda = \beta$. Therefore, the corresponding Chvátal-Gomory cut $(A^\top \lambda)^\top x \leq \lfloor b^\top \lambda \rfloor$ is equivalent to $c^\top x \leq \lfloor \beta \rfloor$.

Conversely, let $(A^\top\lambda)^\top x \le \lfloor b^\top\lambda\rfloor$ be a Chvátal-Gomory cut. By construction, $(A^\top\lambda)^\top x \le b^\top\lambda$ is valid for P, and thus

$$P \cap \{x \in \mathbb{R}^d :\ (A^\top\lambda)^\top x \ge \lfloor b^\top\lambda\rfloor + 1\} = \emptyset,$$

as required. □

2.2 Chvátal closure

Let $P^{(1)}$ be the set obtained as the intersection of $P = \{x \in \mathbb{R}^d : Ax \le b\}$ with all Chvátal-Gomory cuts, i.e.,

$$P^{(1)} = \bigcap_{\lambda \in \mathbb{R}^m_+ : A^\top\lambda \in \mathbb{Z}^d} \left\{x \in \mathbb{R}^d :\ (A^\top\lambda)^\top x \le \lfloor b^\top\lambda\rfloor\right\}.$$

Here, this set $P^{(1)}$ is the **Chvátal closure** of P. As there exist infinitely many Chvátal-Gomory cuts, the Chvátal closure is defined by infinitely many linear inequalities. In fact, we will show that the Chvátal closure is a polyhedron, which means that it is defined by finitely many linear inequalities. In other words, all but a finite set of Chvátal-Gomory cuts are redundant.

Theorem 16.2. *Let $P = \{x \in \mathbb{R}^d : Ax \le b\}$ be a rational polyhedron. Then a Chvátal-Gomory cut is implied by an inequality of the form $(A^\top\mu)^\top x \le \lfloor b^\top\mu\rfloor$ where $A^\top\mu \in \mathbb{Z}^d$ and $0 \le \mu < 1$ and inequalities defining P.*

Proof. As P is a rational polyhedron, we may assume that A and b have integer entries. Consider a Chvátal-Gomory cut $(A^\top\lambda)^\top x \le \lfloor b^\top\lambda\rfloor$ with $\lambda \ge 0$. Now take

$$\mu = \lambda - \lfloor\lambda\rfloor.$$

Then $0 \le \mu < 1$. Moreover,

$$A^\top\mu = A^\top\lambda - A^\top\lfloor\lambda\rfloor \in \mathbb{Z}^d$$

because $A^\top\lambda \in \mathbb{Z}^d$ and $A, \lfloor\lambda\rfloor$ have integer entries. Therefore, $(A^\top\mu)^\top x \le \lfloor b^\top\mu\rfloor$ is a Chvátal-Gomory cut, and $(A^\top\lfloor\lambda\rfloor)^\top x \le b^\top\lfloor\lambda\rfloor$ is valid for P. Adding these two inequalities, we obtain

$$\left(A^\top\mu + A^\top\lfloor\lambda\rfloor\right)^\top x \le \lfloor b^\top\mu\rfloor + b^\top\lfloor\lambda\rfloor.$$

Since A and b have integer entries,

$$A^\top\mu + A^\top\lfloor\lambda\rfloor = A^\top\lambda$$
$$\lfloor b^\top\mu\rfloor + b^\top\lfloor\lambda\rfloor = \lfloor b^\top\mu + b^\top\lfloor\lambda\rfloor\rfloor = \lfloor b^\top\lambda\rfloor.$$

Then the inequality is equivalent to

$$(A^\top\lambda)^\top x \le \lfloor b^\top\lambda\rfloor.$$

Therefore, the Chvátal-Gomory cut with multiplier λ is implied by the cut with multiplier μ and valid inequalities $Ax \le b$ for P. □

With Theorem 16.2, we can now prove that the Chvátal closure is a polyhedron.

Theorem 16.3 (Schrijver [Sch80])**.** *The Chvátal closure $P^{(1)}$ is a polyhedron.*

Proof. By Theorem 16.2, we only need to consider $(A^\top\mu)^\top x \le \lfloor b^\top\mu\rfloor$ where $A^\top\mu \in \mathbb{Z}^d$ and $0 \le \mu < 1$. Here, the set

$$\left\{\mu \in \mathbb{R}^m :\ A^\top\mu \in \mathbb{Z}^d,\ 0 \le \mu < 1\right\}$$

is finite. Therefore, the Chvátal closure is obtained by applying a finite set of Chvátal-Gomory cuts. □

2.3 Modular arithmetic and Gomory's fractional cuts

Assume that

$$S = \left\{ x \in \mathbb{Z}_+^d : \sum_{j=1}^d a_j x_j = a_0 \right\}.$$

Let p be a positive integer, and assume that

$$a_j = r_j + q_j p, \quad j = 0, 1, \ldots, d$$

where $0 \leq r_j < p$ and $q_j \in \mathbb{Z}$. Then $\sum_{j=1}^d a_j x_j = a_0$ can be rewritten as

$$\begin{aligned}\sum_{j=1}^d r_j x_j &= r_0 + \left(q_0 p - \sum_{j=1}^d q_j p \right) \\ &= r_0 + \left(q_0 - \sum_{j=1}^d q_j \right) p\end{aligned}$$

Then it follows that

$$S \subseteq \left\{ x \in \mathbb{Z}_+^d : \sum_{j=1}^d r_j x_j = r_0 + kp \text{ where } k \in \mathbb{Z} \right\}.$$

Since $r_1, \ldots, r_d$ are all nonnegative and $x \geq 0$,

$$\sum_{j=1}^d r_j x_j \geq 0.$$

We know that if $x \in S$, then the left-hand side has a value of the form $r_0 + kp$. What is the smallest nonnegative value that $r_0 + kp$ can take? It is attained when $k = 0$, in which case the value is r_0. Therefore, it follows that

$$\sum_{j=1}^d r_j x_j \geq r_0$$

is valid for S. Remember that p is an arbitary positive integer. In particular, we may choose $p = 1$. Then the corresponding inequality is

$$\sum_{j=1}^d (a_j - \lfloor a_j \rfloor) x_j \geq a_0 - \lfloor a_0 \rfloor.$$

This is a **Gomory's fractional cut**.

Remark 16.4. To generate a Gomory's fractional cut for $S = \left\{ x \in \mathbb{Z}_+^d : Ax \leq b \right\}$, we first add slack variables to make constraints equalities. More precisely, we add nonnegative variables s to have

$$Ax + s = b.$$

Then any equation

$$(A^\top \lambda)^\top x + \lambda^\top s = b^\top \lambda$$

can be used to produce a fractional cut.

Example 16.5. Consider the following system of constraints.

$$\begin{aligned} -x_1 + 2x_2 &\le 4 \\ 5x_1 + x_2 &\le 20 \\ -2x_1 - 2x_2 &\le -7 \\ x_1, x_2 &\in \mathbb{Z}_+ \end{aligned}$$

By adding slack variables, we obtain

$$\begin{aligned} -x_1 + 2x_2 + s_1 &= 4 \\ 5x_1 + x_2 + s_2 &= 20 \\ -2x_1 - 2x_2 + s_3 &= -7 \\ x_1, x_2 &\in \mathbb{Z}_+ \\ s_1, s_2, s_3 &\ge 0 \end{aligned}$$

Let us multiply $\lambda = (-1/11, 2/11, 0)$ to the equality constraints. Then we deduce that

$$x_1 - \frac{1}{11}s_1 + \frac{2}{11}s_2 = \frac{36}{11}.$$

Then the corresponding Gomory's fractional cut is given by

$$\frac{10}{11}s_1 + \frac{2}{11}s_2 \ge \frac{3}{11}.$$

In the original space, this is equivalent to

$$\frac{10}{11}(4 + x_1 - 2x_2) + \frac{2}{11}(20 - 5x_1 - x_2) \ge \frac{3}{11}.$$

This inequality gives us

$$x_2 \le \frac{7}{2}.$$

Let $P = \left\{x \in \mathbb{R}^d_+ : Ax \le b\right\}$ be a polyhedron where A and b have integer entries. Let the **Gomory fractional closure** of P is defined as the set obtained from P after applying all Gomory's fractional cuts.

Theorem 16.6. *The Chvátal closure of P is identical to the Gomory fractional closure of P.*

Proof. We first show that the Chvátal closure of P is contained in the Gomory fractional closure of P. To do so, it suffices to argue that a Gomory's fractional cut is implied by a Chvátal-Gomory cut. Adding slack variables s, we deduce

$$Ax + s = \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = b.$$

Then a Gomory's fractional cut is obtained from

$$\lambda^\top \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = b^\top \lambda$$

for some $\lambda \ge 0$. Here the equation can be rewritten as

$$(A^\top \lambda)^\top x + \lambda^\top s = b^\top \lambda.$$

Then Gomory's fractional cut is given by

$$\left(A^\top \lambda - \lfloor A^\top \lambda \rfloor\right)^\top x + (\lambda - \lfloor \lambda \rfloor)^\top s \geq b^\top \lambda - \lfloor b^\top \lambda \rfloor.$$

We will show that this is implied by a Chvátal-Gomory cut. Let μ be defined as

$$\mu = \lambda - \lfloor \lambda \rfloor.$$

As $\mu \geq 0$, it follows that

$$(A^\top \mu)^\top x + \mu^\top s \geq \mu^\top b \tag{16.1}$$

is valid for P. Moreover, we know that

$$\lfloor A^\top \mu \rfloor^\top x \leq \lfloor b^\top \mu \rfloor \tag{16.2}$$

is the Chvátal-Gomory cut associated with μ. Subtracting (16.2) from (16.1), we obtain

$$(A^\top \mu - \lfloor A^\top \mu \rfloor)^\top x + \mu^\top s \geq b^\top \mu - \lfloor b^\top \mu \rfloor. \tag{16.3}$$

Note that

$$\begin{aligned} A^\top \mu - \lfloor A^\top \mu \rfloor &= A^\top \lambda - A^\top \lfloor \lambda \rfloor - \lfloor A^\top \lambda - A^\top \lfloor \lambda \rfloor \rfloor \\ &= A^\top \lambda - A^\top \lfloor \lambda \rfloor - \lfloor A^\top \lambda \rfloor + A^\top \lfloor \lambda \rfloor \\ &= A^\top \lambda - \lfloor A^\top \lambda \rfloor \end{aligned}$$

where the second equality holds because A has integer entries. Moreover,

$$\begin{aligned} b^\top \mu - \lfloor b^\top \mu \rfloor &= b^\top \lambda - b^\top \lfloor \lambda \rfloor - \lfloor b^\top \lambda - b^\top \lfloor \lambda \rfloor \rfloor \\ &= b^\top \lambda - b^\top \lfloor \lambda \rfloor - \lfloor b^\top \lambda \rfloor + b^\top \lfloor \lambda \rfloor \\ &= b^\top \lambda - \lfloor b^\top \lambda \rfloor \end{aligned}$$

where the second equality holds because b has integer entries. Therefore, (16.3) is equivalent to the Gomory's fractional cut associated with λ.

Conversely, we next prove that the Gomory fractional closure of P is contained in the Chvátal closure of P. We will argue that a Chvátal-Gomory cut is implied by a Gomory's fractional cut. By Theorem 16.2, we may focus on a Chvátal-Gomory cut of the form

$$\lfloor A^\top \mu \rfloor^\top x \leq \lfloor b^\top \mu \rfloor$$

where $0 \leq \mu < 1$. Since $0 \leq \mu < 1$, the corresponding Gomory's fractional cut is given by

$$\left(A^\top \mu - \lfloor A^\top \mu \rfloor\right)^\top x + \mu^\top s \geq b^\top \mu - \lfloor b^\top \mu \rfloor.$$

We subtract the Gomory's fractional cut from

$$(A^\top \mu)^\top x + \mu^\top s = b^\top \mu.$$

Then we deduce that

$$\lfloor A^\top \mu \rfloor^\top x \leq \lfloor b^\top \mu \rfloor,$$

which is precisely the Chvátal-Gomory cut. □

References

[Sch80] A. Schrijver. On cutting planes. *Annals of Discrete Mathematics*, 9:291–296, 1980. 16.3