1 Outline

In this lecture, we study

- Chvátal-Gomory cuts,
- odd-set inequalities for matching,
- Chvátal's rounding procedure.

2 Chvátal-Gomory cuts

Starting with this lecture, we discuss **general-purpose cutting planes** for integer programming in-depth. What is a general-purpose cutting plane? Given a polyhedron $P = \{x \in \mathbb{R}^d_+ : Ax \leq b\}$ and the set of integer solutions $S = P \cap \mathbb{Z}^d$, we want to find an inequality $c^\top x \leq d$ such that

- $c^\top x \leq d$ is valid for S, i.e., every point $z \in S$ satisfies $c^\top z \leq d$,
- $c^\top x \leq d$ may be violated by some point in P.

Here, a general-purpose cutting plane is a prodcedure of generating such inequalities no matter which values of A and b are given. The first is the so-called **Chvátal-Gomory cuts**.

Ralph Gomory [Gom58] discovered the first finitely convergent cutting plane algorithm for solving pure integer linear programs. The cuts used within his algorithms are called **Gomory's fractional cuts**. In fact, the terminology refers to a "procedure" of generating valid inequalities/cuts for a given (pure) integer program. That said, one may generate and apply Gomory's fractional cuts for any pure integer linear programs. For this reason, Gomory's fractional cuts are general-purpose cutting planes.

Vašek Chvátal [Chv73] later studied a cut-generation scheme that unifies some of the existing valid inequalities for combinatorial optimization problems such as the matching problem and the stable set problem. Although the initial focus was on combinatorial optimization problems, the cut-generation scheme can be applied to any pure integer linear programs. In the paper, Chvátal proved that the cuts obtained from his cut-generation scheme are essentially equivalent to Gomory's fractional cuts. For this reason, the generated cuts are now called **Chvátal-Gomory cuts**.

Chvátal-Gomory cuts are prevalent in the discrete optimization literature. Many fundamental classes of facet-defining inequalities for combinatorial optimization problems are Chvátal-Gomory cuts, e.g., odd set inequalities for the matching problem and odd circuit inequalities for the stable set problem. Chvátal-Gomory cuts are computationally effective for solving integer linear programs in practice [FL07, BCD$^+$08], and Chvátal-Gomory cuts for nonlinear integer programs have also been studied [ÇI05].

Then, what are Chvátal-Gomory cuts? How do we obtain them? Today, we will see combinatorial, rounding, arithmetic, and geometric arguments to obtain Chvátal-Gomory cuts.

2.1 Combinatorial argument

Consider the matching problem. Let $G = (V, E)$ be an undirected graph, and the integer programming formulation for the matching problem is given as follows.

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e \leq 1, \quad v \in V \\ & x_e \in \{0,1\}, \quad e \in E \end{aligned}$$

where

- w_e is the weight of edge $e \in E$,
- $\delta(v) = \{e \in E : v \text{ is adjacent to } e\}$.

Here, we want to generate valid inequalities for

$$S = \left\{ x \in \{0,1\}^E : \sum_{e \in \delta(v)} x_e \leq 1, \quad v \in V \right\}.$$

Let $U \subseteq V$ be a subset of the vertex set with an odd number of vertices. Then look at the set of

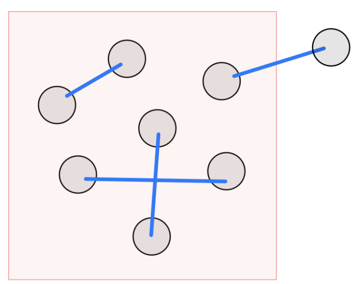

Figure 15.1: Odd cardinality subset

edges that are fully contained in U. Then the following inequality is satisfied by S.

$$\sum_{e \in E(U)} x_e \leq \frac{|U| - 1}{2}$$

where $E(U)$ is the set of edges fully contained in U. We call this inequality an **odd-set inequality**. Why is the odd-set inequality valid? Note that the left-hand side $\sum_{e \in E(U)} x_e$ counts the maximum number of edges from $E(U)$ a maching can take. Here, a matching takes an edge in $E(U)$ means that two vertices in U are matched. Note that $|U|$ is odd, and by parity, at least one vertex always remains unmatched. Equivalently, at most $|U| - 1$ vertices in U can be matched by a matching. Hence, $E(U)$ contains at most $(|U| - 1)/2$ edges in a matching.

Chvátal [Chv73] developed the following systematic argument to derive odd-set inequalities. First, take the inequalities $\sum_{e \in \delta(v)} x_e \leq 1$ for all vertices $v \in U$, and sum them up:

$$\sum_{v \in U} \sum_{e \in \delta(v)} x_e \leq |U|.$$

Here, in the left-hand side, x_e appears twice if e is fully contained in u, and x_e appears once if only one of e's two ends is contained in U. In other words, x_e appears twice if $e \in E(U)$, and x_e appears once if $e \in \delta(U)$ where

$$\delta(U) = \{e \in E : \text{ one end of } e \text{ is in } U \text{ while the other end of } e \text{ is not in } U\}.$$

This implies that the inequality is equivalent to

$$2\sum_{e\in E(U)} x_e + \sum_{e\in\delta(U)} x_e \le |U|.$$

Now let us divide each side by 2.

$$\sum_{e\in E(U)} x_e + \frac{1}{2}\sum_{e\in\delta(U)} x_e \le \frac{|U|}{2}.$$

Here, as $x_e \ge 0$ for all $e \ge E$, the inequality implies that

$$\sum_{e\in E(U)} x_e + \lfloor\frac{1}{2}\rfloor\sum_{e\in\delta(U)} x_e = \sum_{e\in E(U)} x_e \le \frac{|U|}{2}.$$

Finally, we know that the left-hand side is an integer, so rounding down the right-hand side still preserves validity.

$$\sum_{e\in E(U)} x_e \le \lfloor\frac{|U|}{2}\rfloor = \frac{|U|-1}{2}.$$

As a result, we deduce the odd-set inequality for subset U.

In fact, the odd set inequalities describe the **matching polytope**.

Theorem 15.1 (Edmonds [Edm65]). *For the matching problem,*

$$\mathrm{conv}(S) = \left\{x \in \mathbb{R}^E_+ : \sum_{e\in\delta(v)} x_e \le 1, \quad \forall v \in V, \quad \sum_{e\in E(U)} x_e \le \frac{|U|-1}{2}, \quad \forall U \subseteq V \text{ odd}\right\}.$$

Here, $\mathrm{conv}(S)$ *is called the* ***matching polytope****.*

2.2 Chvátal's integer rounding procedure

Given $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, let $\lfloor v\rfloor$ denote the vector $(\lfloor v_1\rfloor, \ldots, \lfloor v_d\rfloor)$ obtained after rounding down each component of v. Let $P = \{x \in \mathbb{R}^d_+ : Ax \le b\}$ and $S = P \cap \mathbb{Z}^d$. Here, Chvátal's rounding procedure is as follows.

1. Take $\lambda \in \mathbb{R}^m_+$ and $\lambda^\top Ax \le \lambda^\top b$ valid for P.
2. Round down the left-hand side to obtain $\lfloor A^\top\lambda\rfloor^\top x \le b^\top\lambda$. This inequality is valid because $\lambda^\top Ax \le \lambda^\top b$ and $x \ge 0$.
3. Round down the right-hand side to obtain $\lfloor A^\top\lambda\rfloor^\top x \le \lfloor b^\top\lambda\rfloor$. This inequality is valid because $x \in \mathbb{Z}^d$ and $\lfloor A^\top\lambda\rfloor \in \mathbb{Z}^d$.

In general, we consider polyhedron $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ that is not necessarily contained in $\mathbb{R}^d_+$ and $S = P \cap \mathbb{Z}^d$. For general polyhedra, Chvátal's rounding procedure works as follows.

1. Take $\lambda \in \mathbb{R}^m_+$ such that $A^\top \lambda \in \mathbb{Z}^d$. Then $(A^\top \lambda)^\top x \leq b^\top \lambda$ is valid for P.
2. Round down the right-hand side to obtain $(A^\top \lambda)^\top x \leq \lfloor b^\top \lambda \rfloor$. This inequality is valid because $x \in \mathbb{Z}^d$ and $A^\top \lambda \in \mathbb{Z}^d$.

In fact, the rounding procedure for the nonnegative case, i.e., $P \subseteq \mathbb{R}^d_+$, is a special case. When P is given by $P = \{x \in \mathbb{R}^d_+ : Ax \leq b\}$, the constraint system is given by

$$Ax \leq b, \ -x \leq 0.$$

Recall that we took a multiplier vector λ to obtain the corresponding Chvátal-Gomory cut

$$\lfloor A^\top \lambda \rfloor^\top x \leq b^\top \lambda.$$

Note that this inequality is equivalent to

$$\lambda^\top Ax + (A^\top \lambda - \lfloor A^\top \lambda \rfloor)^\top (-x) \leq b^\top \lambda.$$

This is equivalent to take multipliers $(\lambda, A^\top \lambda - \lfloor A^\top \lambda \rfloor)$ for system $Ax \leq b, -x \leq 0$.

An inequality that can be obtained from Chvátal's rounding procedure is a Chvátal-Gomory cut. Note that the multiplier vector λ can be arbitrary. In particular, there are infinitely many choices for λ, thereby leading to infinitely many Chvátal-Gomory inequalities.

References

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