IE 631 Integer Programming **KAIST, Spring 2023**
Lecture #12: Minimum cost flow, Total unimodularity April 6, 2023
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1 Outline

In this lecture, we study

- the minimum cost flow problem,
- totally unimodular matrices.

2 Minimum cost flow problem

A **directed graph** $D = (N, A)$ consists of a set of **nodes** N and a set of **arcs** $A \subseteq N \times N$. A directed graph is often referred to as a **network**. Here, an arc is an ordered pair of two nodes. Figure 12.1 shows a network over 6 nodes with 9 arcs in total. The node set N and the arc set A

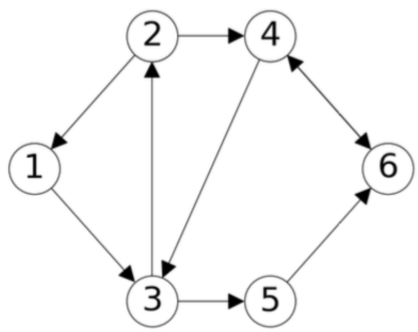

Figure 12.1: Network over 6 nodes

are given by

$$N = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad A = \{(1, 3), (2, 1), (2, 4), (3, 2), (3, 5), (4, 3), (4, 6), (5, 6), (6, 4)\}.$$

One of the most general network flow models is the **minimum cost flow model**. Here, think of **flow** as some quantity, such as water, electricity, money, and products, that needs to be routed around the network. Given a directed graph $D = (N, A)$, we define the following components of the problem.

Decisions: We use variable x_{ij} for each arc $(i, j) \in A$ to decide how much flow travels across arc (i, j). To encode **discrete quantities**, such as the number of products sent from one city to another, we impose that

$$x_{ij} \in \mathbb{Z}, \quad \forall (i, j) \in A.$$

Flow bound constraints: There are upper and lower bounds on how much flow an arc can accommodate. For each arc $(i, j) \in A$,

$$\ell_{ij} \le x_{ij} \le u_{ij}$$

for some $\ell_{ij}, u_{ij} \geq 0$. Here, u_{ij} can take $+\infty$, in which case we simply write $x_{ij} \geq \ell_{ij}$ without the upper bound. ℓ_{ij} is often set to 0. By defining vectors ℓ and u that collect the lower and upper bounds of arc flows, we can summarize the constraints as

$$\ell \leq x \leq u.$$

Flow balance constraints: For each node $i \in N$ and the vector x of flow values on the arcs, the **outflow** is defined as the amount of flow out of the node i:

$$\text{outflow}(i; x) := \sum_{j \in N:(i,j) \in A} x_{ij}.$$

The **inflow** is defined as the amount of flow into the node i:

$$\text{inflow}(i; x) := \sum_{k \in N:(k,i) \in A} x_{ki}.$$

The **net supply** of node i is the difference of the outflow and the inflow:

$$\text{net-supply}(i; x) = \text{outflow}(i; x) - \text{inflow}(i; x) = \sum_{j \in N:(i,j) \in A} x_{ij} - \sum_{k \in N:(k,i) \in A} x_{ki}.$$

Each node i satisfies a flow balance constraint, given by

$$\text{net-supply}(i; x) = \sum_{j \in N:(i,j) \in A} x_{ij} - \sum_{k \in N:(k,i) \in A} x_{ki} = b_i.$$

Here, each node i is either a supply or a demand node, given by a parameter b_i.

- If $b_i > 0$, i.e., the net supply is positive, then i is a **supply node**.
- If $b_i < 0$, i.e., the net supply is negative, then i is a **demand node**.
- If $b_i = 0$, i.e., the net supply is zero, then i is a **transhipment node**.

Objective: Directing one unit of flow from node i to node j incurs a cost of c_{ij}. Then the objective is to minimize the total cost.

$$\text{minimize} \quad \sum_{(i,j) \in A} c_{ij} x_{ij}.$$

To summarize, the minimum cost flow problem over network $D = (N, A)$ is modeled as the following integer linear program.

$$\begin{aligned} \min \quad & \sum_{(i,j) \in A} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j \in N:(i,j) \in A} x_{ij} - \sum_{k \in N:(k,i) \in A} x_{ki} = b_i, \quad i \in N \\ & \ell \leq x \leq u \\ & x_{ij} \in \mathbb{Z}, \quad (i,j) \in A. \end{aligned}$$

Given a directed graph $D = (N, A)$, the **node-arc incidence matrix** M has $|N|$ rows corresponding to the nodes and $|A|$ columns corresponding to the arcs. The entries of M is given by

$$m_{i,(k,j)} = \begin{cases} 1, & \text{if } k = i, \\ -1, & \text{if } j = i, \\ 0, & \text{if } k \neq i \text{ and } j \neq i \end{cases}$$

for any $i \in N$ and $(k, j) \in A$. We often refer to M as a **network matrix**. For example, the directed graph over 6 nodes has the incidence matrix given as the following table.

$m_{i,(k,j)}$	$(1,3)$	$(2,1)$	$(2,4)$	$(3,2)$	$(3,5)$	$(4,3)$	$(4,6)$	$(5,6)$	$(6,4)$
1	1	-1							
2		1	1	-1					
3	-1			1	1	-1			
4			-1			1	1		-1
5					-1			1	
6							-1	-1	1

The incidence matrix has the following properties.

- Entries are -1, 0, and $+1$ only.
- Each column has only two nonzero entries, $+1$ and -1.
- The column for arc (k, j) has $+1$ in row k and -1 in row j.
- Adding up all rows of M, we obtain a row of all zeros, i.e., $\mathbf{1}^\top Mx = 0$.
- Adding up any subset of rows of M, we obtain a vector with entries $-1, 0, 1$ only.

Let b be the vector with entries b_i for $i \in N$. Then we may write the flow balance constraints as

$$Mx = b.$$

The ith row of this matrix equation is

$$\sum_{(k,j)\in A} m_{i,(k,j)} x_{kj} = b_i.$$

Here, the left-hand side is given by

$$\begin{aligned} \sum_{(k,j)\in A} m_{i,(k,j)} x_{kj} &= \sum_{(k,j)\in A:k=i} m_{i,(k,j)} x_{kj} + \sum_{(k,j)\in A:j=i} m_{i,(k,j)} x_{kj} \\ &= \sum_{(k,j)\in A:k=i} x_{kj} - \sum_{(k,j)\in A:j=i} x_{kj} \\ &= \sum_{j\in N:(i,j)\in A} x_{ij} - \sum_{k\in N:(k,i)\in A} x_{ki}. \end{aligned}$$

Therefore, $Mx = b$ indeed collects the set of flow balance constraints.

Remark 12.1. If $Mx = b$ is feasible, then

$$0 = \mathbf{1}^\top Mx = \mathbf{1}^\top b = \sum_{i \in N} b_i.$$

Therefore, if $\sum_{i \in N} b_i \neq 0$, then $Mx = b$ is infeasible.

With the incidence matrix, the minimum cost flow model can be written as

$$\begin{aligned} \min \quad & \sum_{(i,j) \in A} c_{ij} x_{ij} \\ \text{s.t.} \quad & Mx = b \\ & \ell \leq x \leq u \\ & x_{ij} \in \mathbb{Z}, \quad (i,j) \in A. \end{aligned}$$

Theorem 12.2. *Let M be the node-arc incidence matrix of a network. Consider the LP relaxation*

$$\begin{aligned} \min \quad & \sum_{(i,j) \in A} c_{ij} x_{ij} \\ s.t. \quad & Mx = b \\ & \ell \leq x \leq u \end{aligned}$$

Suppose that b, ℓ, u have only integer entries. Then there exists an optimal solution x^* to the LP relaxation that has only integer entries.

3 Totally unimodular matrices

Let M be an $m \times d$ matrix. A **submatrix** of M is a matrix that consists of the entries in a subset of rows and a subset of columns. For example, we take nodes $\{2, 4, 6\}$ and columns $\{(2,4), (4,6), (6,4)\}$.

$m_{i,(k,j)}$	(1, 3)	(2, 1)	(2, 4)	(3, 2)	(3, 5)	(4, 3)	(4, 6)	(5, 6)	(6, 4)
1	1	−1							
2		1	1	−1					
3	−1			1	1	−1			
4			−1			1	1		−1
5					−1			1	
6							−1	−1	1

Then the corresponding submatrix of the node-arc incidence matrix is given by

	(2, 4)	(4, 6)	(6, 4)
2	1	0	0
4	−1	1	−1
6	0	−1	1

A **square submatrix** of M is a submatrix of M that is a square matrix, i.e., the number of rows and that of columns are equal.

A matrix M is **totally unimodular** if every square submatrix of M has determinant $-1, 0, 1$ only. Note that each entry itself is an 1×1 square submatrix, so if M is totally unimodular, all its entries are $-1, 0, 1$ only.

Theorem 12.3 (Hoffman and Kruskal [HK56])**.** *Consider a linear program given by*

$$\begin{aligned} \min \quad & c^\top x \\ s.t. \quad & Px \le b. \end{aligned}$$

If P is totally unimodular and b has integer entries only, then there exists an optimal solution x^* to the linear program that has only integer entries.

Theorem 12.4. *Let M be the node-incidence matrix of a network, and let I be the identity matrix that has the same number of columns as M. Then*

$$\begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix}$$

is totally unimodular.

Note that by definition, if a matrix is totally unimodular, then all its submatrices are totally unimodular. In particular, the node-arc incidence matrix M itself is totally unimodular.

We now prove Theorem 12.2.

Proof of Theorem 12.2. The LP relaxation of the minimum cost flow model is given by

$$\begin{aligned} \min \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\ \text{s.t.} \quad & Mx = b, \\ & \ell \le x \le u. \end{aligned}$$

We may write the constraints as

$$\begin{aligned} Mx &\le b \\ -Mx &\le -b \\ x &\le u \\ -x &\le -\ell \end{aligned}$$

Then the constraints can be taken into the following matrix inequality form.

$$\begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix} x \le \begin{bmatrix} b \\ -b \\ u \\ -\ell \end{bmatrix}.$$

By Theorem 12.4, the resulting constraint matrix is totally unimodular. As $b, -b, u, -\ell$ have all integer entries, it follows from Theorem 12.3 that there is an optimal solution x^* that has integer entries only. □

How do we prove Theorem 12.4 and that a network matrix is totally unimodular? An **equitable column bicoloring** of a matrix M is partition of its columns into two sets, say red and blue columns, such that the sum of the red columns minus the sum of the blue columns is a vector of entries $-1, 0, 1$ only. An **equitable row bicoloring** can be similarly defined.

Theorem 12.5 (Ghouila-Houri [GH62]). *A matrix M is totally unimodular if and only if every column submatrix of M admits an equitable bicoloring.*

Here, a column submatrix is a submatrix that keeps all rows. Note that M is totally unimodular if and only if its transpose $M^\top$ is totally unimodular.

Theorem 12.6. *Any network matrix M is totally unimodular.*

Proof. Since M is totally unimodular if and only if $M^\top$ is totally unimodular, we may apply Thoerem 12.5 to M^T. Then, it follows that M is totally unimodular if and only if every row submatrix of M admits an equitable row bicoloring. Note that every column of M has at most one $+1$ and at most one -1. Then for any row submatrix M' of M, we assign only the red color to the rows. Note that adding up all rows of M', which corresponds to a subset of rows of M, we obtain a vector of entries $-1, 0, 1$ only. This means that coloring all rows of M' red gives rise to an equitable row bicoloring. □

References

[GH62] A. Ghouila-Houri. Caractérisation des matrices totalement unimodulaires. *Comptes Rendus Hebdomadaires des Sc éances de l'Académie des Sciences (Paris)*, 254:1192–1194, 1962. 12.5

[HK56] A.J. Hoffman and J.B. Kruskal. Integral boundary points of convex polyhedra. In Kuhn H.W. and Tucker A.W., editors, *Linear inequalities and related systems, Ann. Math. Studies*, volume 38, pages 223–246. 1956. 12.3