1 Outline

In this lecture, we study

- Faces,
- Redundant inequalities and facets,
- Extreme points and extreme rays.

2 Recession cone and lineality space examples

Example 11.1. Consider a polyhedron given by

$$P = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 \geq 1, \ x_2 \geq 2, \ x_1 + x_2 \leq 3\right\}.$$

Note that P consists of a single point $(x_1, x_2) = (1, 2)$ as shown in Figure 11.1. Moreover, $(1, 2)$

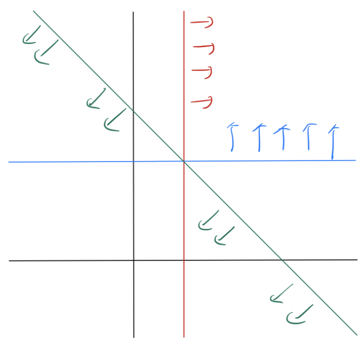

Figure 11.1: First example

satisfies the three constraints with equality. Therefore, the three constraints are implicit equalities of P.

What is the dimension of P? As P contains a single point, we can infer that the dimension of P is 0. Let us confirm this by looking at its affine hull. Remember that the affine hull of P is defined simply by the set of implicit equalities. Hence,

$$\text{aff}(P) = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 \geq 1, \ x_2 \geq 2, \ x_1 + x_2 \leq 3\right\} = \{(1, 2)\}.$$

The system of implicit equalities is given by

$$A^= = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Here, the rank of $A^=$, denoted $\text{rank}(A^=) = 2$. Then

$$\dim(P) = 2 - \text{rank}(A^=) = 2 - 2 = 0.$$

Example 11.2. Consider a polyhedron given by

$$P = \left\{(x_1, x_2) \in \mathbb{R}^2 : \; x_1 - x_2 \leq -3\right\}.$$

Note that P is a half-space as shown in Figure 11.2. The recession cone of P is given by

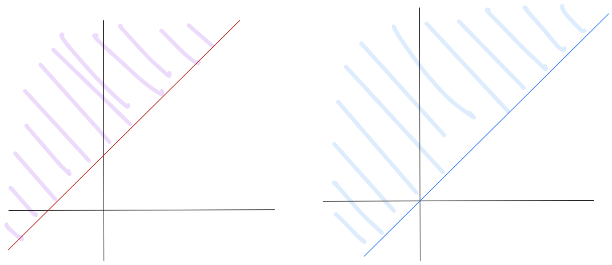

Figure 11.2: Second example and its recession cone & lineality space

$$\operatorname{rec}(P) = \left\{(x_1, x_2) \in \mathbb{R}^2 : \; x_1 - x_2 \leq 0\right\}.$$

Moreover, the lineality space of P is given by

$$\operatorname{lin}(P) = \left\{(x_1, x_2) \in \mathbb{R}^2 : \; x_1 - x_2 = 0\right\}.$$

In Figure 11.2, the blue line corresponds to the lineality space of P, and the half-space, including the blue line, is the recession of P. Moreover, P has no implicit equality, so $\operatorname{aff}(P) = \mathbb{R}^2$. This means that the dimension of P is 2.

3 Faces

Let $P = \left\{x \in \mathbb{R}^d : \; Ax \leq b\right\}$ be a polyhedron. Remember that an inequality $\alpha^\top x \leq \beta$ is valid for P if every point in P satisfies the inequality. Given a valid inequality $\alpha^\top x \leq \beta$,

$$P \cap \left\{x \in \mathbb{R}^d : \; \alpha^\top x = \beta\right\} = \left\{x \in \mathbb{R}^d : \; Ax \leq b, \; \alpha^\top x = \beta\right\}$$

is called the **face** induced by inequality $\alpha^\top x \leq \beta$. A face F is **proper** if $F \neq \emptyset$ and $F \neq P$.

In Figure 11.3, there is a polyhedron in $\mathbb{R}^2$. There are three distinct hyperplanes, each of which

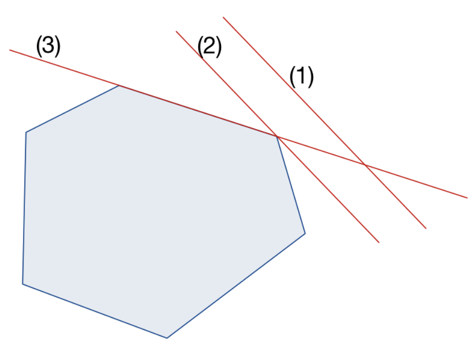

Figure 11.3: 2-dimensional polyhedron and faces

gives rise to a valid inequality.Valid inequality (1) defines an empty face. Valid inequality (2) defines a 0-dimensional face because the hyperplane intersects the polyhedron at a single point. Lastly, valid inequality (3) defines an 1-dimensional face.

In fact, we can define a face without a valid inequality. Suppose that the system $Ax \leq b$ consists of m inequalities $a_i^\top x \leq b_i$ for $i \in [m]$.

Proposition 11.3. *If F is a nonempty face of P, then there exists $M \subseteq [m]$ such that*

$$F = \left\{ x \in \mathbb{R}^d : \ a_i^\top x = b_i \ \ \forall i \in M, \ \ a_j^\top x \leq b_j \ \ \forall j \in [m] \setminus M \right\}.$$

Consequently, F is a polyhedron.

Proof. Suppose that F is induced by a valid inequality $\alpha^\top x \leq \beta$. Then

$$\begin{aligned} \beta \quad = \quad \max \quad & \alpha^\top x \\ \text{s.t.} \quad & Ax \leq b. \end{aligned}$$

Moreover, F is the set of optimal solutions to this linear program. As F is nonempty, the linear program is bounded and feasible. By strong duality, it follows that

$$\begin{aligned} \beta \quad = \quad \min \quad & \beta^\top y \\ \text{s.t.} \quad & A^\top y = \alpha \\ & y \geq 0. \end{aligned}$$

Let y^* be an optimal solution to the dual linear program. Then we define M as

$$M = \{ i \in [m] : \ y_i^* > 0 \}.$$

By complementary slackness, we deduce that

$$F = \left\{ x \in \mathbb{R}^d : \ a_i^\top x = b_i \ \ \forall i \in M, \ \ a_j^\top x \leq b_j \ \ \forall j \in [m] \setminus M \right\},$$

as required. □

4 Redundant inequalities and facets

Let $P = \left\{ x \in \mathbb{R}^d : \ Ax \leq b \right\}$ be a polyhedron. Suppose that $Ax \leq b$ consists of m linear inequalities $a_i^\top x \leq b_i$ for $i \in [m]$. We say that an inequality $a_k^\top x \leq b_k$ is **redundant** if the system $a_i^\top x \leq b_i$ for $i \in [m] \setminus \{k\}$ induces the same polyhedron P. Hence, we may remove a redudant inequality without changing the polyhedron.

Proposition 11.4. *Suppose that the face of P induced by $a_k^\top x \leq b_k$ has dimension at most* $\dim(P) - 2$ *where* $\dim(P)$ *is the dimension of P. Then $a_k^\top x \leq b_k$ is redundant.*

Proof. We prove the contrapositive of the statement. Suppose that $a_k^\top x \leq b_k$ is a non-redundant inequality. If $a_k^\top x \leq b_k$ is an implicit equality, then the face induced by $a_k^\top x \leq b_k$ is P itself, which has dimension $\dim(P)$. Thus we may assume that $a_k^\top x \leq b_k$ is not an implicit equality. Suppose that $M^= \subseteq [m]$ corresponds to the set of implicit equalities and $M^<$ collects the other inequalities. Recall that there exists $\hat{x}$ such that

$$a_i^\top \hat{x} = b_i \ \ \forall i \in M^= \quad \text{and} \quad a_i^\top \hat{x} < b_i \ \ \forall i \in M^<.$$

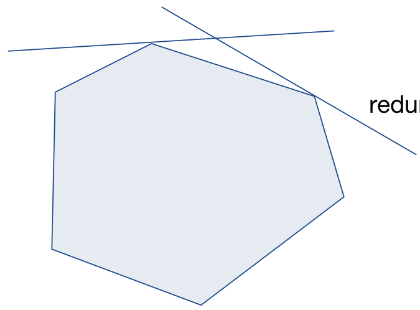

Figure 11.4: 2-dimensional polyhedron and redundant inequalities

In words, $\hat{x}$ strictly satisfies all constraints that are not an implicit equality.
As $a_k^\top x \le b_k$ is not redundant, there exists $\bar{x}$ such that

$$a_i^\top \bar{x} \le b_i \quad \forall i \in [m] \setminus \{k\} \quad \text{and} \quad a_k^\top \bar{x} > b_k.$$

Then it follows that

$$a_i^\top \bar{x} = b_i \quad \forall i \in M^=, \quad a_i^\top \bar{x} \le b_i \quad \forall i \in M^< \setminus \{k\}, \quad a_k^\top \bar{x} > b_k.$$

Next we consider a line segment connecting $\bar{x}$ and $\hat{x}$. For any $\lambda \in [0,1]$, we have

$$a_i^\top (\lambda \bar{x} + (1-\lambda)\hat{x}) = b_i \quad \forall i \in M^=, \quad a_i^\top (\lambda \bar{x} + (1-\lambda)\hat{x}) < b_i \quad \forall i \in M^< \setminus \{k\}.$$

More importantly, as $a_k^\top \hat{x} < b_k$ and $a_k^\top \bar{x} > b_k$, there exists $\lambda^* \in [0,1]$ such that

$$a_k^\top (\lambda^* \bar{x} + (1-\lambda^*)\hat{x}) = b_k.$$

In fact, such λ is given by

$$\lambda^* = \frac{b_k - a_k^\top \hat{x}}{a_k^\top (\bar{x} - \hat{x})} = \frac{b_k - a_k^\top \hat{x}}{(a_k^\top \bar{x} - b_k) + (b_k - a_k^\top \hat{x})} \in (0,1).$$

Therefore,

$$x^* = \lambda^* \bar{x} + (1-\lambda^*)\hat{x}$$

satisfies

$$a_i^\top x^* = b_i \quad \forall i \in M^=, \quad a_i^\top x^* < b_i \quad \forall i \in M^< \setminus \{k\}, \quad a_k^\top x^* = b_k.$$

This means that $M^= \cup \{k\}$ is precisely the set of implicit equalities in the face

$$F = \left\{ x \in \mathbb{R}^d : \; Ax \le b, \; a_k^\top x = b_k \right\}.$$

As F has just one more implicit equality than P, it follows that $\dim(F) \ge \dim(P) - 1$. $\square$

A **facet** of polyhedron P is a non-empty face of dimension $\dim(P) - 1$.

Proposition 11.5. *For each facet F of polyhedron P, there exists an inequality $a_k^\top x \le b_k$ which is not an implicit equality and induces the facet F.*

We say that a valid inequality $\alpha^\top x \le \beta$ is **facet-defining** for P if the face induced by the inequality is a facet of P.

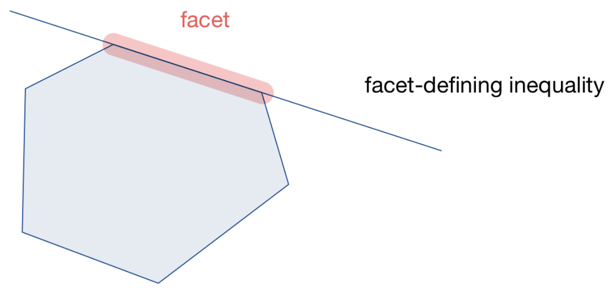

Figure 11.5: 2-dimensional polyhedron and a facet

Example 11.6. Let $G = (V, E)$ be a graph. Then the stable set polytope is given by

$$\text{stab}(G) = \text{conv}\left\{x \in \{0,1\}^V :\ x_u + x_v \leq 1 \text{ for all } (u,v) \in E\right\}.$$

Let K be a maximal clique. We learned that the clique inequality

$$\sum_{v \in K} x_v \leq 1$$

is valid for stab(G). In fact, we will show that the clique inequality induces a facet of stab(G). To prove this, it suffices to show that the face

$$F = \text{stab}(G) \cap \left\{x \in \mathbb{R}^V :\ \sum_{v \in K} x_v = 1\right\}.$$

has dimension $\dim(\text{stab}(G)) - 1$.

First of all, $\emptyset$ and all singletons $\{v\}$ for $v \in V$ are stable sets. That means stab(G) contains 0 and e_v for $v \in V$ where e_v is the unit vector that has 1 in the position of v. Hence stab(G) contains $|V| + 1$ affinely independent points, and therefore, $\dim(\text{stab}(G)) = |V|$.

To show that F has dimension $|V| - 1$, we exhibit $|V|$ affinely independent points in stab(G). For each $v \in K$, $\{v\}$ is a stable set, and moreover, $|K \cap \{v\}| = |\{v\}| = 1$. For each $u \in V \setminus K$, there is a vertex $w \in K$ such that (u, w) is not an edge. This is because if $u \in V \setminus K$ is connected to all vertices in K, then we may add u to K, and the resulting set would be a clique. However, this violates the assumption that K is maximal. In such case, as (u, w) is not an edge, $\{u, w\}$ is a stable set such that $|K \cap \{u, w\}| = |\{w\}| = 1$. In fact,

$$e_v \quad \text{for } v \in K, \quad e_u + e_w \quad \text{for all } u \in V \setminus K \text{ and some } w \in K \text{ such that } (u,w) \text{ is not an edge}$$

are $|V|$ affinely independent points in stab(G).

5 Extreme points and extreme rays

A face of dimension 0 is called a **vertex** or an **extreme point** of polyhedron P.

Theorem 11.7. *Polyhedron P has a vertex if and only if P is pointed, i.e.,* $\text{lin}(P) = \{0\}$.

The next theorem characterizes a vertex of a pointed polyhedron P in terms of its inequality system.

Theorem 11.8. *Let $P = \{x \in \mathbb{R}^d : \ Ax \leq b\}$ be a pointed polyhedron. Then the following statements are equivalent.*

(i) *$\bar{x}$ is a vertex of P.*

(ii) *$\bar{x}$ satisfies d linearly independent inequalities in $Ax \leq b$ at equality.*

(iii) *$\bar{x}$ cannot be expressed as a proper convex combination of two distinct points in P.*

Here, a ***proper*** *convex combination of two points u, v mean $\lambda u + (1 - \lambda)v$ for some $0 < \lambda < 1$.*

A face of dimension 1 is called an **edge** of polyhedron P.

Theorem 11.9. *Let $P = \{x \in \mathbb{R}^d : \ Ax \leq b\}$ be a pointed polyhedron. Then the following statements are equivalent.*

(i) *$\bar{r}$ is an extreme ray of P.*

(ii) *$\bar{r}$ satisfies $d - 1$ linearly independent inequalities in $Ax \leq 0$ at equality.*

(iii) *$\bar{r}$ cannot be expressed as a proper convex combination of two distinct rays in* $\text{rec}(P)$.