1 Outline

In this lecture, we study

- Minkowski-Weyl theorem for general polyhedra.
- recession cone and lineality space.
- implicit equalities and affine hull.

2 Minkowski-Weyl theorem for general polyhedra

A set $P \subseteq \mathbb{R}^d$ is a **polyhedron** if it is defined by a **finite** number of linear inequalities, i.e.

$$P = \{x \in \mathbb{R}^d : \; Ax \leq b\}.$$

Hence, a polyhedron is a finite intersection of half-spaces. A polyhedron is **rational** if it is defined by

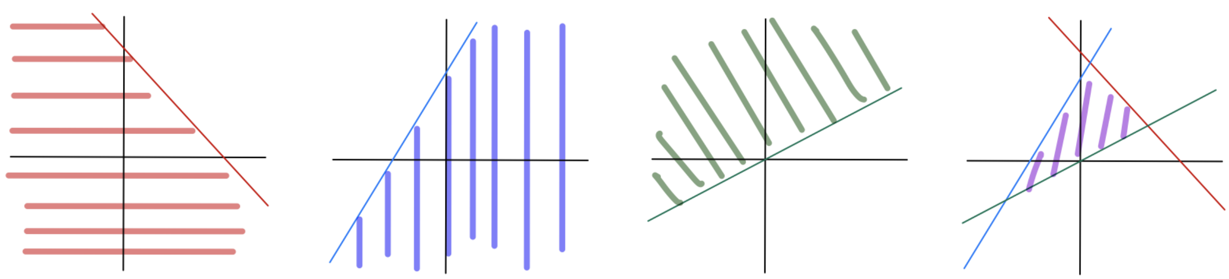

Figure 10.1: Polyhedron defined by three inequalities

a system of linear inequalities where all coefficients and right-hand sides are rational. A polyhedral cone is by definition a polyhedron.

Given a polyhedron $P = \{x \in \mathbb{R}^d : \; Ax \leq b\}$, we can associate a polyhedral cone given by

$$C_P = \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \; Ax - by \leq 0, \; y \geq 0\right\}.$$

Then we have

$$P = \left\{x \in \mathbb{R}^d : \; (x, 1) \in C_P\right\}.$$

Given two sets $Q, C \subseteq \mathbb{R}^d$, **Minkowski sum** of Q and C is defined as

$$Q + C = \left\{x = u + v \in \mathbb{R}^d : \; u \in Q \; v \in C\right\}.$$

Based on these, we may prove the following theorem for polyhedra.

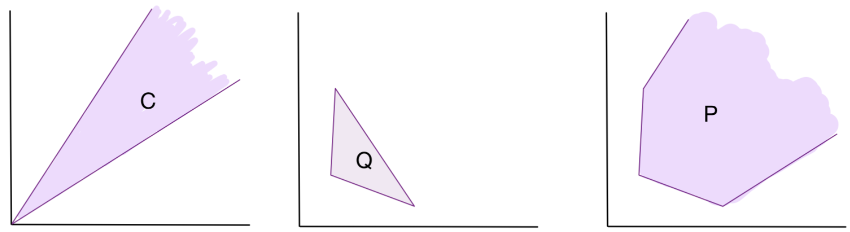

Figure 10.2: Polyhedron as the Minkowski sum of a polytope and a polyhedral cone

Theorem 10.1 (Minkowski-Weyl theorem)**.** *A set $P \subseteq \mathbb{R}^d$ is a polyhedron if and only if*

$$P = \text{conv}(v^1, \dots, v^p) + \text{cone}(r^1, \dots, r^q)$$

for some vectors $v^1, \dots, v^p$ and $r^1, \dots, r^q$.

Proof. **(⇒)** Let $P = \{x : \mathbb{R}^d : Ax \leq b\}$ be a polyhedron. Then consider the associated cone $C_P = \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \; Ax - by \leq 0, \; y \geq 0\right\}$. By Minkowski-Weyl theorem for cones, there exist vectors $(u^1, w^1), \dots, (u^k, w^k) \in \mathbb{R}^d \times \mathbb{R}$ such that

$$C_P = \text{cone}\left(\begin{pmatrix} u^1 \\ w^1 \end{pmatrix}, \dots, \begin{pmatrix} u^k \\ w^k \end{pmatrix}\right).$$

By the definition of C_P, we have $w^1, \dots, w^k \geq 0$. Then each w^i is strictly positive or equal to 0. For each vector (u^i, w^i) with $w^i > 0$, we divide it by w^i. Then C_P can be rewritten as

$$\begin{aligned} C_P &= \text{cone}\left(\begin{pmatrix} v^1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} v^p \\ 1 \end{pmatrix}, \begin{pmatrix} r^1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} r^q \\ 0 \end{pmatrix}\right) \\ &= \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \; \exists \lambda \geq 0, \mu \geq 0 \text{ s.t. } x = \sum_{i=1}^{p} \lambda_i v^i + \sum_{j=1}^{q} \mu_j r^j, \; y = \sum_{i=1}^{p} \lambda_i \right\} \end{aligned}$$

Since $P = \left\{x \in \mathbb{R}^d : \; (x, 1) \in C_P\right\}$, it follows that

$$P = \text{conv}(v^1, \dots, v^p) + \text{cone}(r^1, \dots, r^q).$$

(⇐) Let C_P be defined as

$$C_P = \text{cone}\left(\begin{pmatrix} v^1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} v^p \\ 1 \end{pmatrix}, \begin{pmatrix} r^1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} r^q \\ 0 \end{pmatrix}\right).$$

By Minkowski-Weyl theorem for cones, we know that C_P is a polyhedral cone, and therefore, C_P can be written as

$$C_P = \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \; Ax - by \leq 0\right\}.$$

Note that $P = \left\{x \in \mathbb{R}^d : \; (x, 1) \in C_P\right\}$, in which case, we have

$$P = \{x \in \mathbb{R}^d : \; Ax \leq b\},$$

as required. ∎

A set $P \subseteq \mathbb{R}^d$ is a **polytope** if it is a polyhedron and bounded, i.e., $P \subseteq [-M, M]^d$ for some sufficiently large $M > 0$.

Corollary 10.2 (Minkowski-Weyl theorem for polytopes)**.** *A set $P \subseteq \mathbb{R}^d$ is a polytope if and only if*

$$P = \text{conv}(v^1, \ldots, v^p)$$

for some vectors $v^1, \ldots, v^p$.

Proof. **(⇒)** By Theorem 10.1, as P is a polyhedron, $P = \text{conv}(v^1, \ldots, v^p) + \text{cone}(r^1, \ldots, r^q)$ for some vectors $v^1, \ldots, v^p$ and $r^1, \ldots, r^q$. Here, since P is bounded, we have $q = 0$ or $r^1 = \cdots = r^q = 0$, in which case $P = \text{conv}(v^1, \ldots, v^p)$.

(⇐) By Theorem 10.1, P is a polyhedron. Moreover, since P is the convex hull of a finite number of vectors, P is bounded. Therefore, P is a polytope. □

3 Recession cone and lineality space

Given a nonempty polyhedron P, a **ray** of P is a vector r such that

$$x + \lambda r \in P \quad \text{for all } x \in P \text{ and } \lambda \geq 0.$$

The **recession cone** of P is the set of all rays of P, i.e.,

$$\text{rec}(P) = \left\{ r \in \mathbb{R}^d : \ x + \lambda r \in P \quad \text{for all } x \in P \text{ and } \lambda \geq 0 \right\}.$$

Lemma 10.3. rec(P) *is a convex cone.*

Proof. Let $r^1, r^2 \in \text{rec}(P)$. Then $x + \lambda r^1, x + \lambda r^2 \in P$ for any $x \in P$ and $\lambda \geq 0$. Since P is convex, for any $0 \leq \alpha \leq 1$, we have

$$x + \lambda \left(\alpha r^1 + (1 - \alpha) r^2 \right) = \alpha(x + \lambda r^1) + (1 - \alpha)(x + \lambda r^2) \in P.$$

This implies that $\alpha r^1 + (1 - \alpha) r^2 \in \text{rec}(P)$, and therefore, rec$(P)$ is a convex cone. □

A ray r of polyhedron P is called an **extreme ray** of P if it cannot be written as a convex combination of two distinct rays of P.

The **lineality space** of P is defined as

$$\text{lin}(P) = \left\{ r \in \mathbb{R}^d : \ x + \lambda r \in P \quad \text{for all } x \in P \text{ and } \lambda \in \mathbb{R} \right\}.$$

Hence, the lineality space is the set of all vectors r such that both r and $-r$ are rays of P.

Lemma 10.4. $\text{lin}(P) = \text{rec}(P) \cap -\text{rec}(P)$.

Proof. Note that $r \in \text{lin}(P)$ if and only if $r \in \text{rec}(P)$ and $-r \in \text{rec}(P)$. Moreover, $-r \in \text{rec}(P)$ if and only if $r \in -\text{rec}(P)$. Therefore, $r \in \text{lin}(P)$ if and only if $r \in \text{rec}(P)$ and $r \in -\text{rec}(P)$. □

Moreover,

Lemma 10.5. lin(P) *is a linear subspace.*

Proof. Let $r^1, r^2 \in \text{lin}(P)$. Then $x + 2\lambda r^1, x + 2\lambda r^2 \in P$ for any $x \in P$ and $\lambda \in \mathbb{R}$. Since P is convex, we have

$$x + \lambda\left(r^1 + r^2\right) = \frac{1}{2}(x + 2\lambda r^1) + \frac{1}{2}(x + 2\lambda r^2) \in P.$$

This implies that $r^1 + r^2 \in \text{lin}(P)$. Moreover, for any $\alpha \in \mathbb{R}$, we know that $x + \lambda\alpha r^1 \in P$. This implies that $\alpha r^1 \in \text{lin}(P)$. Therefore, $\text{lin}(P)$ is a linear subspace. □

We say that polyhedron P is **pointed** if its lineality space is **trivial**, i.e., $\text{lin}(P) = \{0\}$.

Proposition 10.6. *Let $P = \left\{x \in \mathbb{R}^d : \ Ax \leq b\right\}$ be a polyhedron such that*

$$P = \text{conv}(v^1, \ldots, v^p) + \text{cone}(r^1, \ldots, r^q)$$

for some vectors $v^1, \ldots, v^p$ and $r^1, \ldots, r^q$[1]*. Then the recession cone of P is given by*

$$\text{rec}(P) = \left\{x \in \mathbb{R}^d : \ Ax \leq 0\right\} = \text{cone}(r^1, \ldots, r^q).$$

Moreover, the lineality space is given by

$$\text{lin}(P) = \left\{x \in \mathbb{R}^d : \ Ax = 0\right\} = \text{cone}(r^1, \ldots, r^q) \cap \text{cone}(-r^1, \ldots, -r^q).$$

Proof. We first show that $\text{rec}(P) = \left\{x \in \mathbb{R}^d : \ Ax \leq 0\right\}$. Let r satify $Ar \leq 0$. Then for any $x \in P$ and $\lambda \geq 0$, we have $A(x + \lambda r) \leq Ax \leq b$, implying that $x + \lambda r \in P$. Hence $r \in \text{rec}(P)$. Let r be such that $Ar \not\leq 0$. Then $(Ar)_i > 0$ for some component i. Then for any $x \in P$, there exists a sufficiently large $\lambda > 0$ such that $(Ax)_i + \lambda(Ar)_i > 0$. In this case, $Ax + \lambda Ar = A(x + \lambda r) \not\leq 0$, and therefore, $r \notin \text{rec}(P)$.

Next we show that $\text{rec}(P) = \text{cone}(r^1, \ldots, r^q)$. If $r \in \text{cone}(r^1, \ldots, r^q)$, then $x + \lambda r \in P$ for any $x \in P$ and $\lambda \geq 0$. Therefore, $r \in \text{rec}(P)$. Let $r \in \text{rec}(P)$. Then for any $x \in \text{conv}(v^1, \ldots, v^p)$, we have $x + \lambda r \in P$ for $\lambda \geq 0$. Since $\text{conv}(v^1, \ldots, v^p)$ is bounded, we must have $r \in \text{cone}(r^1, \ldots, r^q)$.

By the previous lemma, we have that $\text{lin}(P) = \text{rec}(P) \cap -\text{rec}(P)$. Note that

$$-\text{rec}(P) = \left\{x \in \mathbb{R}^d : \ Ax \geq 0\right\} = \text{cone}(-r^1, \ldots, -r^q),$$

as required. □

4 Implicit equalities and affine hull

Consider a polyhedron $P = \left\{x \in \mathbb{R}^d : \ Ax \leq b\right\}$ where $Ax \leq b$ consists of linear inequalities $a_i^\top x \leq b_i$ for $i \in [m]$. We say that $a_i^\top x \leq b_i$ is an **implicit equality** if

$$P \subseteq \left\{x \in \mathbb{R}^d : \ a_i^\top x = b_i\right\}.$$

In words, $a_i^\top x \leq b_i$ is an implicit equality if every point in P satisfies it with equality. If there is a point x in P such that $a_i^\top < b_i$, i.e., x satisfies the inequality strictly, then $a_i^\top x \leq b_i$ is not an implicit equality.

Let $A^= x \leq b^=$ be the subsystem of $Ax \leq b$ that collects all implicit equalities, and let $A^< x \leq b^<$ collect the other inequalities in $Ax \leq b$.

[1]These vectors exist due to Minkowski-Weyl theorem

Lemma 10.7. *There is a point $\bar{x} \in P$ that satisfies $A^{<}\bar{x} < b^{<}$, i.e., $\bar{x}$ satisfies all inequalities $A^{<}x \leq b^{<}$ strictly.*

Proof. Suppose that $A^{<}x \leq b^{<}$ is given by $a_i^\top x \leq b_i$ for $i \in I$ where I is some subset of $[m]$. For each $i \in I$, as $a_i^\top x \leq b_i$ is not an implicit equality, there exists some $x^i \in P$ such that $a_i^\top x^i < b_i$. Morever, as x^i is point in P, it satisfies $Ax^i \leq b$. Then we take a convex combination of x^i for $i \in I$ given by

$$\bar{x} = \frac{1}{|I|}\sum_{i \in I} x^i.$$

Then for any $i \in I$, we have $a_i^\top \bar{x} < b_i$. □

Theorem 10.8. *The affine hull of polyhedron P is given by*

$$\mathrm{aff}(P) = \left\{x \in \mathbb{R}^d : \; A^{=}x = b^{=}\right\} = \left\{x \in \mathbb{R}^d : \; A^{=}x \leq b^{=}\right\}.$$

In particular, $\dim(P) = d - \mathrm{rank}(A^{=})$.

Proof. It is straightforward that

$$\mathrm{aff}(P) \subseteq \left\{x \in \mathbb{R}^d : \; A^{=}x = b^{=}\right\} \subseteq \left\{x \in \mathbb{R}^d : \; A^{=}x \leq b^{=}\right\}.$$

Then it suffices to argue that

$$\left\{x \in \mathbb{R}^d : \; A^{=}x \leq b^{=}\right\} \subseteq \mathrm{aff}(P).$$

Let $\hat{x}$ satisfy $A^{=}\hat{x} \leq b^{=}$. By Lemma 10.7, there exists $\bar{x} \in P$ such that $A^{<}\bar{x} < b^{<}$. Then for some sufficiently small $\epsilon > 0$, we have

$$A^{<}(\bar{x} + \epsilon(\hat{x} - \bar{x})) = A^{<}\bar{x} + \epsilon A^{<}(\hat{x} - \bar{x}) \leq b^{<}.$$

Moreover,

$$A^{=}(\bar{x} + \epsilon(\hat{x} - \bar{x})) = (1-\epsilon)A^{=}\bar{x} + \epsilon A^{=}\hat{x} \leq b^{=}.$$

Let $\tilde{x} = \bar{x} + \epsilon(\hat{x} - \bar{x})$. Then it follows that $\tilde{x} \in P$. Note that as $\bar{x}, \tilde{x} \in P$, the line going through $\bar{x}$ and $\tilde{x}$ is contained in the affine hull $\mathrm{aff}(P)$. Moreover,

$$\hat{x} = \frac{1}{\epsilon}\tilde{x} - \frac{1-\epsilon}{\epsilon}\hat{x}.$$

Here, the coefficients sum up to 1, and therefore, $\hat{x}$ is on the line going through $\bar{x}$ and $\tilde{x}$. Therefore, $\hat{x} \in \mathrm{aff}(P)$. □

Polyhedron $P = \left\{x \in \mathbb{R}^d : \; Ax \leq b\right\}$ is **full-dimensional** if $\dim(P) = d$, in which case, the system $Ax \leq b$ does not involve an implicit equality.

Example 10.9. Let us consider the **assignment polytope**, given by

$$P = \left\{ x \in \mathbb{R}^{n^2} : \begin{array}{rl} \sum_{j=1}^n x_{ij} = 1, & i = 1, \dots, n \\ \sum_{i=1}^n x_{ij} = 1, & j = 1, \dots, n \\ x_{ij} \geq 0, & i, j = 1, \dots, n \end{array} \right\}.$$

We may prove that $\dim(P) = n^2 - 2n + 1$.

Note that the system defining P has $2n$ equality constraints. Let $Ax = 1$ denote the system that collects the $2n$ equality constraints. We will show that $\text{rank}(A) = 2n - 1$. Note that

$$\underbrace{\sum_{i=1}\sum_{j=1} x_{ij}}_{\text{aggregating the first set of equalities}} \quad - \quad \underbrace{\sum_{j=1}\sum_{i=1} x_{ij}}_{\text{aggregating the second set}} \quad = 0.$$

This implies that A is not of full row rank, and therefore, $\text{rank}(A) \le 2n - 1$. Next, consider the column submatrix of A associated with variables x_{1i} for $i = 1, \ldots, n$ and x_{ii} for $i = 2, \ldots, n$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \\ 1 & & & & & & \\ & 1 & & & 1 & & \\ & & 1 & & & 1 & \\ & & & 1 & & & 1 \end{bmatrix}$$

The submatrix has $2n - 1$ columns which are linearly independent. Therefore, $\text{rank}(A) \ge 2n - 1$. This implies that $\text{rank}(A) = 2n - 1$. Then it follows from Theorem 10.8 that $\dim(P) = n^2 - \text{rank}(A) = n^2 - 2n + 1$, as required.