IE 539 Convex Optimization Final Exam

Fall 2024

Out: 18st December 2024

Due: 18th December 2024 at 11:59pm

Instructions

- Submit a PDF document with your solutions through the assignment portal on KLMS by the due date. Please ensure that your name and student ID are on the front page.
- Late submissions will **not** be accepted except in extenuating circumstances. Special consideration should be applied for in this case.
- It is **required** that you typeset your solutions in LaTeX. Handwritten solutions will not be accepted.
- Spend some time ensuring your arguments are **coherent** and your solutions **clearly** communicate your ideas.

Question:	1	2	3	Total
Points:	40	10	50	100

Let $f_1 : \mathbb{R}^d \to \mathbb{R}$ and $f_2 : \mathbb{R}^d \to \mathbb{R}$ be β-smooth convex functions. Then take function $f : \mathbb{R}^d \to \mathbb{R}$ as the point-wise maximum of f_1 and f_2, i.e.,

$$f(x) = \max\{f_1(x), f_2(x)\}$$

for all $x \in \mathbb{R}^d$. We are interested in solving the optimization problem:

$$\min_{x\in\mathbb{R}^d} \quad f(x).$$

To solve this, we consider the following algorithm.

1. Let $x_1 \in \mathbb{R}^d$ be an initial solution.
2. For $t = 1, \dots, T$, we repeat the following.
   - Given $x_t \in \mathbb{R}^d$, we take two functions $g_{1,t} : \mathbb{R}^d \to \mathbb{R}$ and $g_{2,t} : \mathbb{R}^d \to \mathbb{R}$ as
   $$g_{1,t}(x) = f_1(x_t) + \nabla f_1(x_t)^\top (x - x_t) + \frac{\beta}{2}\|x - x_t\|_2^2,$$
   $$g_{1,t}(x) = f_2(x_t) + \nabla f_2(x_t)^\top (x - x_t) + \frac{\beta}{2}\|x - x_t\|_2^2.$$
   - Take function $g_t : \mathbb{R}^d \to \mathbb{R}$ as $g_t(x) = \max\{g_{1,t}(x), g_{2,t}(x)\}$ for $x \in \mathbb{R}^d$.
   - Then we take x_{t+1} as $x_{t+1} = \arg\min_{x\in\mathbb{R}^d} g_t(x)$.

In the following questions, we walk through the convergence analysis of this algorithm.

1. (40 points) Prove that for each t and for all $x \in \mathbb{R}^d$,
$$g_t(x) - \frac{\beta}{2}\|x - x_t\|_2^2 \geq g_t(x_{t+1}) + \beta(x_t - x_{t+1})^\top (x - x_t) + \frac{\beta}{2}\|x_t - x_{t+1}\|_2^2.$$

2. (10 points) Prove that for each t,
$$f(x_{t+1}) \leq f(x_t) - \frac{\beta}{2}\|x_t - x_{t+1}\|_2^2.$$

3. (50 points) Prove that for each t,
$$f(x_{T+1}) - f(x^*) \leq \frac{\beta}{2T}\|x_1 - x^*\|_2^2.$$