IE 539 Convex Optimization Assignment 5

Fall 2024

Out: 4th December 2024

Due: 15th December 2024 at 11:59pm

Instructions

- Submit a PDF document with your solutions through the assignment portal on KLMS by the due date. Please ensure that your name and student ID are on the front page.
- Late assignments will be subject to a penalty. Special consideration should be applied for in this case.
- It is **required** that you typeset your solutions in LaTeX. Handwritten solutions will not be accepted.
- Spend some time ensuring your arguments are **coherent** and your solutions **clearly** communicate your ideas.

Question:	1	2	3	4	Total
Points:	25	20	15	40	100

1. Given points $v_1, \ldots, v_n \in \mathbb{R}^d$, consider the location problem

$$\min_{x\in\mathbb{R}^d} \max_{k\in[n]} \|x - v_k\|_2.$$

This problem can be reformulated as the following second-order cone program (SOCP).

$$\begin{aligned}\min_{x\in\mathbb{R}^d} \max_{k\in[n]} \|x - v_k\|_2 &= \min_{x,t} \{t : \|x - v_k\|_2 \le t \ \forall k \in [n]\} \\ &= \min_{x,t} \{t : (x - v_k, t) \in \mathcal{L}_{d+1} \ \forall k \in [n]\}\end{aligned}$$

where $\mathcal{L}_{d+1} = \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \|x\|_2 \le t\right\}$ is the Lorentz cone.

(a) (10 points) Derive the dual of the SOCP.

(b) (5 points) Prove that the strong duality holds.

(c) (10 points) Show that the optimal location is in the convex hull of $\{v_1, \ldots, v_n\}$.

2. (a) (10 points) Derive the dual of

$$\text{minimize} \quad f(x) + \|Ax - b\|.$$

Here, $\|\cdot\|$ is an arbitrary norm.

(b) (10 points) Derive the dual of

$$\begin{aligned}\text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in C_1 \cap \cdots \cap C_k\end{aligned}$$

where $C_1, \ldots, C_k$ are some convex sets in $\mathbb{R}^d$. You may use the indicator functions of the sets and their Fenchel conjugates.

3. (15 points) Let $f(x) = \|x\|_1 : \mathbb{R}^d \to \mathbb{R}$. Prove that the Moreau-Yosida smoothing of f is given by

$$f_\eta(x) = \sum_{i=1}^d \frac{1}{\eta} L_\eta(x_i)$$

where

$$L_\eta(c) = \begin{cases} \eta|c| - \eta^2/2, & \text{if } |c| \ge \eta, \\ |c|^2/2, & \text{if } |c| \le \eta. \end{cases}$$

4. Consider

$$h(\mu) = f^*(-A^\top \mu) + g^*(\mu)$$

for some convex functions f and g.

(a) (15 points) Prove that

$$\mu_{t+1} = \mu_t - \eta_t g_t \quad \text{where } g_t \in \partial h(\mu_t)$$

if and only if

$$\mu_{t+1} = \mu_t + \eta(Ax_t - y_t)$$

for some

$$\begin{aligned} x_t &\in \arg\min_x \left\{f(x) + \mu_t^\top Ax\right\}, \\ y_t &\in \arg\min_y \left\{g(y) - \mu_t^\top y\right\}. \end{aligned}$$

(b) (15 points) Prove that

$$\mu_{t+1} = \operatorname{prox}_{\eta h}(\mu_t)$$

if and only if

$$\mu_{t+1} = \mu_t + \eta(Ax_t - y_t)$$

for some

$$(x_t, y_t) \in \arg\min_{(x,y)} \left\{f(x) + g(y) + \mu_t^\top (Ax - y) + \frac{\eta}{2}\|Ax - y\|_2^2\right\}.$$

(c) (10 points) Prove that $\operatorname{prox}_{\eta h}(\mu) = \mu - \eta \nabla h_\eta(\mu)$.