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1 Outline

In this lecture, we consider

- conic duality theorems
- second-order cone programming

2 Conic duality theorem

In the last lecture, we learned that a *conic program* is an optimization problem is defined with a pointed and closed convex cone K with a nonempty interior:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax - b \in K. \end{aligned} \tag{CP}$$

When $K = \mathbb{R}^n_+$, (CP) reduces to linear program

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \geq b. \end{aligned} \tag{LP}$$

We know that the dual of the linear program (LP) is given by

$$\begin{aligned} \text{maximize} \quad & b^\top y \\ \text{subject to} \quad & A^\top y = c \\ & y \geq 0. \end{aligned} \tag{dual-LP}$$

Moreover, the dual of (CP) can be derived by the following procedure.

(1) Take x such that $Ax - b \in K$ and $y \in K^*$. Then $y^\top (Ax - b) \geq 0$, and therefore,

$$y^\top Ax \geq y^\top b.$$

(2) If $y \in K^*$ further satisfies $A^\top y = c$, then

$$c^\top x = y^\top Ax \geq y^\top b = b^\top y.$$

(3) Then

$$\begin{aligned} \text{maximize} \quad & b^\top y \\ \text{subject to} \quad & A^\top y = c \\ & y \in K^* \end{aligned} \tag{dual-CP}$$

provides a lower bound on the value of (CP). Here, (dual-CP) is the dual conic program of (CP).

Remark 7.1. Here, what is the intuition behind (dual-CP)? What we argued while deriving (dual-CP) is the following. Take a feasible solution x to (CP). Then we have $Ax - b \in K$. Now suppose that we have $y \in K^*$, where K^* is the dual cone of K, such that $A^\top y = c$. Then we saw that

$$c^\top x \geq b^\top y.$$

Note that this holds for any pair of (x, y) satisfying the condition. In particular, what this implies that $b^\top y$ provides a lower bound on the optimum value of (CP), as we may minimize $c^\top x$ over all x satisfying $Ax - b \in K$.

That being said, then what does (dual-CP) do? Well, we saw that for any y with $A^\top y = c$ and $y \in K^*$, $b^\top y$ provides a lower bound. What is the strongest possible lower bound obtained that way? (dual-CP) computes it.

We have shown that the value of the conic program (CP) is lower bounded by the value of its dual, given by (dual-CP). What is striking is that the two values coincide under some conditions! For the case of linear programming, this is known as the strong LP duality theorem. To be precise, the value of (LP) and that of (dual-LP) are the same if (LP) is feasible and bounded. It turns out that we may extend this strong duality result to general conic programs.

We have already observed that the weak duality for LP extends to conic programming. Formally,

Theorem 7.2. *The optimal value of* (dual-CP) *is a lower bound on that of* (CP)*.*

Before we state the strong duality theorem for conic programming, we need to define the notion of *strict feasibility.*

Definition 7.3. We say that a solution x is *strictly feasible* to (CP) if $Ax - b$ belongs to the interior of K. When that is the case, we write it as

$$Ax - b >_K 0 \quad \text{or} \quad Ax >_K 0.$$

Moreover, we say that (CP) is *strictly feasible* if it has a strictly feasible solution.

For example, x is strictly feasible to (LP) if $Ax > b$. Now we are ready to state the following duality result.

Theorem 7.4 (See Theorem 2.4.1 in [BTN01])**.** *The following statements hold for* (CP) *and* (dual-CP)*.*

1. *If* (CP) *is strictly feasible and bounded, then* (dual-CP) *is solvable and the optimal values of* (CP) *and* (dual-CP) *are the same.*
2. *If* (dual-CP) *is strictly feasible and bounded, then* (CP) *is solvable and the optimal values of* (CP) *and* (dual-CP) *are the same.*

We will learn the notion of *Lagrangian duality* later in the course, and the strict feasibility condition is analogous to the *Slater condition.*

3 Second-order cone programming

A *second-order cone program (SOCP)* is an optimization problem of the following form:

$$\begin{aligned} \text{minimize} \quad & f^\top x \\ \text{subject to} \quad & \|A_i x + b_i\|_2 \le c_i^\top x + d_i \quad \text{for } i = 1, \dots, m, \\ & Ex = g. \end{aligned} \tag{SOCP}$$

Exercise 7.5. Prove that (SOCP) is a conic program.

3.1 Example: chance-constrained linear programming

We consider a *chance-constrained program (CCP)* given as follows.

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & \mathbb{P}\left(a^\top x \le b\right) \ge 1 - \epsilon \end{aligned} \tag{CCP}$$

where the constraint vector a is random. The problem is to find a solution x satisfying the constraint $a^\top x \le b$ with probability at least $1 - \epsilon$. For example, we can formulate the portfolio optimization as (CCP).

$$\begin{aligned} \text{minimize} \quad & p^\top x \\ \text{subject to} \quad & \mathbb{P}\left(r^\top x \ge 1 + \alpha\right) \ge 1 - \epsilon, \\ & 1^\top x = 1 \end{aligned}$$

where p_i is the unit price of financial asset $i \in [d]$, and r_i is the random return of asset $i \in [d]$. The goal of the problem is to find a portfolio x whose return is at least $1 + \alpha$ with probability at least $1 - \epsilon$ while minimizing the unit price of the portfolio.

When the coefficient vector a in (CCP) follows the multivariate Gaussian distribution with mean $\bar{a}$ and covariance Σ, we can reformulate it as a second-order cone program. Let $u = a^\top x$. Then u is a random variable with mean $\bar{u} = \bar{a}^\top x$ and variance $\sigma^2 = x^\top \Sigma x$. Then $\mathbb{P}\left(a^\top x \le b\right) \ge 1 - \epsilon$ can be rewritten as

$$\mathbb{P}\left(\frac{u - \bar{u}}{\sigma} \le \frac{b - \bar{u}}{\sigma}\right) \ge 1 - \epsilon. \tag{7.1}$$

Here,

$$\frac{u - \bar{u}}{\sigma}$$

follows the Gaussian distribution with mean 0 and variance 1 whose cumulative distribution function is given by Φ, i.e.

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt.$$

Then (7.1) is equivalent to

$$\Phi\left(\frac{b - \bar{u}}{\sigma}\right) \ge 1 - \epsilon$$

and therefore is equivalent to

$$\frac{b - \bar{u}}{\sigma} = \frac{b - \bar{a}^\top x}{\|\Sigma^{1/2} x\|_2} \ge \Phi^{-1}(1 - \epsilon).$$

Hence, (CCP) can be equivalently written as

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & \bar{a}^\top x + \Phi^{-1}(1-\epsilon)\|\Sigma^{1/2} x\|_2 \le b. \end{aligned}$$

3.2 Example: convex quadratic programming

Recall that a convex quadratic program can be rewritten as follows.

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & t \ge \frac{1}{2} x^\top Q x + p^\top x. \end{aligned}$$

Here, the constraint $t \ge \frac{1}{2} x^\top Q x + p^\top x$ can be written as a second-order cone constraint. Recall that for any positive semidefinite matrix Q, there exists a matrix P such that $Q = P^\top P$. Then $t \ge \frac{1}{2} x^\top Q x + p^\top x$ is equivalent to

$$x^\top P^\top P x \le 2t - 2p^\top x.$$

Note that the left-hand side equals $\|Px\|_2^2$ and that

$$2t - 2p^\top x = (t - p^\top x + 1/2)^2 - (t - p^\top x - 1/2)^2.$$

This implies that $x^\top P^\top P x \le 2t - 2p^\top x$ is equivalent to

$$\|Px\|_2^2 + (t - p^\top x - 1/2)^2 \le (t - p^\top x + 1/2)^2,$$

which is equivalent to

$$\left\| \begin{pmatrix} Px \\ t - p^\top x - 1/2 \end{pmatrix} \right\|_2 \le t - p^\top x + 1/2.$$

Therefore, a convex quadratic program is a second-order cone program.

3.3 Reduction to semidefinite programming

In fact, (SOCP) is an instance of semidefinite programming.

Lemma 7.6. *Let $y \in \mathbb{R}^d$. Then $\|y\|_2 \le s$ is equivalent to*

$$\begin{pmatrix} s & y^\top \\ y & s \cdot I \end{pmatrix} \succeq 0$$

Proof. ($\Leftarrow$) Note that

$$(\|y\|_2, -y^\top) \begin{pmatrix} s & y^\top \\ y & s \cdot I \end{pmatrix} \begin{pmatrix} \|y\|_2 \\ -y \end{pmatrix} = 2(s\|y\|_2^2 - y^\top y \|y\|_2) \ge 0,$$

implying in turn that $s \ge \|y\|_2$.

($\Rightarrow$) Let $u \in \mathbb{R}$ and $v \in \mathbb{R}^d$. Then

$$\begin{aligned} (u, v^\top) \begin{pmatrix} s & y^\top \\ y & s \cdot I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} &= u^2 s + 2u y^\top v + s v^\top v \\ &\ge \|y\|_2 (u^2 + \|v\|_2^2) + 2u y^\top v \\ &\ge \|y\|_2 (u^2 + \|v\|_2^2) - 2|u| \cdot \|y\|_2 \cdot \|v\|_2 \\ &\ge 0 \end{aligned}$$

where the first inequality comes from the assumption that $s \geq \|y\|_2$ and the second and third inequalities are due to the Cauchy-Schwarz inequality. Therefore, the matrix is positive semidefinite as required. □

By Lemma 7.6, any second-order cone constraint can be written as an SDP constraint.

References

[BTN01] Aharon Ben-Tal and Arkadi Nemirovski. *Lectures on Modern Convex Optimization*. Society for Industrial and Applied Mathematics, 2001. 7.4