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1 Outline

In this lecture, we study

- Matrix calculus review
- Convex sets,
- Convex functions.

2 Lipschitz continuity, gradient, and Hessian

We say that a function $f: \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous with respect to norm $\|\cdot\|$ if there exists some nonnegative constant $L \geq 0$ such that

$$|f(x) - f(y)| \le L||x - y||$$
 for all $x, y \in \mathbb{R}^d$.

Here, we say that f is L-Lipschitz with respect to $\|\cdot\|$.

Let $f: \mathbb{R}^d \to \mathbb{R}$ be a function. Let e^i denote the *i*th unit vector. For example, $e^1 = (1, 0, \dots, 0)^{\top}$ and $e^d = (0, \dots, 0, 1)^{\top}$. Then the *i*th partial derivative of f is defined as

$$\frac{\partial f}{\partial x_i}(x) = \lim_{t \to 0} \frac{f(x + te^i) - f(x)}{t}.$$

Thus, the *i*th partial derivative is the directional derivative of f along the *i*th unit direction e^i . If all the partial derivatives of f exist at $x \in \mathbb{R}^d$, then we may define the *gradient* of f at x, given by

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_d}(x)\right)^{\top}.$$

If a function is Lipschitz continuous, then it is continuous and differentiable almost everywhere. However, there is a function that is Lipschitz continuous but not differentiable. For example, f(x) = |x| for $x \in \mathbb{R}$. In addition, $f(x) = |x||_1$ for $x \in \mathbb{R}^d$.

Next, we consider

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \quad \text{for } i,j \in [d]$$

are the second partial derivatives of f. If all the second partial derivatives exist, then we may define the Hessian of f as follows

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_d}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_d \partial x_1}(x) & \frac{\partial^2 f}{\partial x_d \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_d^2}(x) \end{pmatrix}.$$

Moreover, if the second partial derivatives are continuous, then Schwarz's theorem implies that the Hessian is symmetric, i.e.,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x) \quad \text{for every } i, j \in [d].$$

3 A little bit of matrix calculus

Let A be an $n \times d$ matrix and $b \in \mathbb{R}^n$. Let f(x) = g(Ax - b). Then by the chain rule,

$$\nabla f(x) = A^{\mathsf{T}} \nabla g(Ax - b), \quad \nabla^2 f(x) = A^{\mathsf{T}} \nabla^2 g(Ax - b)A.$$

Example 2.1. Consider f(x) = Ax - b. Then $\nabla f(x) = A^{\top}$.

Consider a quadratic function

$$f(x) = x^{\top} Q x + p^{\top} x = \sum_{i=1}^{d} \sum_{j=1}^{d} Q_{ij} x_i x_j + \sum_{j=1}^{d} p_i x_i.$$

Then the gradient of f is given by

$$\nabla f(x) = (Q + Q^{\mathsf{T}})x + p, \quad \nabla^2 f(x) = Q + Q^{\mathsf{T}}.$$

Example 2.2. Consider $f(x) = ||Ax - b||_2^2$. Then

$$\nabla f(x) = 2A^{\mathsf{T}}(Ax - b), \quad \nabla^2 f(x) = 2A^{\mathsf{T}}A.$$

4 Convex sets

4.1 Definition

Definition 2.3. A set $X \subseteq \mathbb{R}^d$ is *convex* if for any $u, v \in X$ and any $\lambda \in [0, 1]$,

$$\lambda u + (1 - \lambda)v \in X$$
.

In words, the line segment joining any two points is entirely contained the set. In Figure 2.1, we have a convex set and a non-convex set.

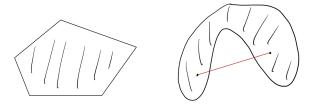


Figure 2.1: A convex set and a nonconvex set

Definition 2.4. Given $v^1, \ldots, v^k \in \mathbb{R}^d$, any linear combination $\lambda_1 v^1 + \cdots + \lambda_k v^k$ is a *convex combination* of v^1, \ldots, v^k if

$$\sum_{i=1}^k \lambda_i = 1 \quad \text{and} \quad \lambda_i \ge 0 \text{ for } i = 1, \dots, k.$$

The convex combination of two distinct points u, v is the line segment $\{\lambda u + (1 - \lambda)v : 0 \le \lambda \le 1\}$ connecting them.

Definition 2.5. The *convex hull* of a set X, denoted conv(X), is the set of all convex combinations of points in X. By definition,

$$\operatorname{conv}(X) = \left\{ \sum_{i=1}^{n} \lambda_i v^i : \sum_{i=1}^{n} \lambda_i = 1, \ \lambda_1, \dots, \lambda_n \ge 0 \right\}.$$

Here, conv(X) is always convex regardless of X. Figure 2.2 shows some examples of taking the convex hull of a set.

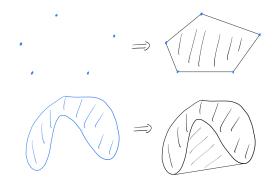


Figure 2.2: A convex set and a nonconvex set

4.2 Cones and affine subspaces

Definition 2.6. A set $C \subseteq \mathbb{R}^d$ is a *cone* if for any $v \in C$ and $\alpha > 0$, we have $\alpha v \in C$. Furthermore, if C is convex, then it is called a *convex cone*.

Note that not all cones are convex.

Definition 2.7. Given $v^1, \ldots, v^k \in \mathbb{R}^d$, any point of the form $\alpha_1 v^1 + \cdots + \alpha v_k v^k$ is a *conic combination* of v^1, \ldots, v^k if $\alpha_1, \ldots, \alpha_k \geq 0$.

In other words, any nonnegative linear combination is a conic combination.

Definition 2.8. The *conic hull* of a set X, denoted cone(X), is the set of all conic combinations of points in X. By definition,

$$\operatorname{conv}(X) = \left\{ \sum_{i=1}^{n} \lambda_{i} v^{i} : \begin{array}{c} n \in \mathbb{N}, \ v^{1}, \dots, v^{n} \in X, \\ \lambda_{1}, \dots, \lambda_{n} \ge 0 \end{array} \right\}.$$

As conv(X), cone(X) is always convex, Figure 2.3 shows an example taking the conic hull of a set in \mathbb{R}^2 .

Lastly, we define the notion of affine subspaces.

Definition 2.9. Given $v^1, \ldots, v^k \in \mathbb{R}^d$, any point of the form $\theta_1 v^1 + \cdots + \theta v_k v^k$ is a affine combination of v^1, \ldots, v^k if $\theta_1 + \cdots + \theta_k = 1$.

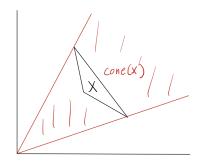


Figure 2.3: Taking the conic hull of a triangle in \mathbb{R}^2

In contrast to covex combinations, affine combinations allow negative multipliers.

Definition 2.10. The affine hull of a set X is the set of all affine combinations of points in X.

The affine hull of X is also referred to as the affine subspace spanned by X. In the previous lecture, we defined the linear subspace spanned by a finite set of vectors, but we can extend the definition to an arbitrary set. The linear hull of a set X is equivalent to the linear subspace spanned by X.

In Figure 2.4, we have a set S of two points in \mathbb{R}^2 . The red line segment is $\operatorname{conv}(S)$, the green line through the two points is the affine subspace spanned by S, the blue cone depicts $\operatorname{cone}(S)$, and lastly, the orange region (in fact, \mathbb{R}^2) is the linear subspace spanned by S.

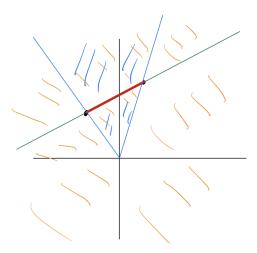


Figure 2.4: Comparing the linear subspace, the affine subspace, the convex hull, and the conic hull

Theorem 2.11. An affine subspace is a translation of a linear subspace. For an affine subspace $V \subseteq \mathbb{R}^d$, there exist matrices A and b such that $V = \{x \in \mathbb{R}^d : Ax = b\}$.

4.3 Examples

We saw that the convex hull and conic hull of a set are convex and that the linear subspace and affine subspace spanned by a set are convex. There are many more examples.

1. Empty set, singletons (sets of the form $\{v\}$),

- 2. Norm ball: $\{x \in \mathbb{R}^d : ||x c|| \le r\}$ where c is the center.
- 3. Ellipsoid: $\{x \in \mathbb{R}^d : (x-c)^\top P(x-c) \leq 1\}$ where P is positive definitie and c is the center.
- 4. Hyperplane: $\{x \in \mathbb{R}^d : a^{\top}x = b\}$ where $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$.
- 5. Half-space: $\{x \in \mathbb{R}^d : a^{\top}x \leq b\}$ where $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$.
- 6. Polyhedron: A polyhedron is a finite intersection of half spaces, $\{x \in \mathbb{R}^d : Ax \leq b\}$ where $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$. Here, $Ax \leq b$ is a short-hand notation for system $a_k^{\top} x \leq b_k$ for $k \in [m]$.
- 7. Polytope: A *polytope* is a polyhedron that is bounded. Equivalently, a polytope is the convex hull of some finite set of vectors.
- 8. Simplex: A set of the form $\{x \in \mathbb{R}^d : 1^{\top}x = 1, x \geq 0\}$, which is equal to the convex hull of e^1, \ldots, e^d , the d-dimensional unit vectors.
- 9. Nonnegative orthant: $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x \ge 0\}.$
- 10. Positive orthant: $\mathbb{R}_{++}^d = \{x \in \mathbb{R}^d : x > 0\}.$

There are examples of convex cones, which are convex as well.

- 1. Norm cone: $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : ||x|| \le t\}$. When $||\cdot||$ is the Euclidean norm, the cone is called the second-order cone.
- 2. Positive semidefinite cone: The set of all positive semidefinite matrices of a fixed dimension.

5 Convex functions

5.1 Definition

Definition 2.12. A function $f: \mathbb{R}^d \to \mathbb{R}$ is *convex* if the domain, denoted dom(f), is convex and for all $x, y \in \text{dom}(f)$, we have

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$$
 for $0 < \lambda < 1$.

In words, function f evaluated at a point between x and y lies below the line segment joining f(x) and f(y).

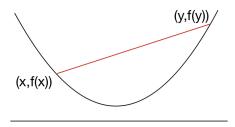


Figure 2.5: Illustration of a convex function in \mathbb{R}^2

Definition 2.13. We say that $f: \mathbb{R}^d \to \mathbb{R}$ is *concave* if -f is convex.

Definition 2.14. A function $f: \mathbb{R}^d \to \mathbb{R}$ is

• strictly convex if dom(f) is convex and for any distinct $x, y \in dom(f)$, we have

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$$
 for $0 < \lambda < 1$.

• strongly convex if $f(x) - \alpha ||x||^2$ is convex for some $\alpha > 0$ and norm $||\cdot||$.

Note that strong convexity implies strict convexity, and strict convexity implies convexity.