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## 1 Outline

In this lecture, we study

- Matrix calculus review
- Convex sets,
- Convex functions.


## 2 Lipschitz continuity, gradient, and Hessian

We say that a function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is Lipschitz continuous with respect to norm $\|\cdot\|$ if there exists some nonnegative constant $L \geq 0$ such that

$$
|f(x)-f(y)| \leq L\|x-y\| \quad \text { for all } x, y \in \mathbb{R}^{d} .
$$

Here, we say that $f$ is $L$-Lipschitz with respect to $\|\cdot\|$.
Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a function. Let $e^{i}$ denote the $i$ th unit vector. For example, $e^{1}=(1,0, \ldots, 0)^{\top}$ and $e^{d}=(0, \ldots, 0,1)^{\top}$. Then the $i$ th partial derivative of $f$ is defined as

$$
\frac{\partial f}{\partial x_{i}}(x)=\lim _{t \rightarrow 0} \frac{f\left(x+t e^{i}\right)-f(x)}{t} .
$$

Thus, the $i$ th partial derivative is the directional derivative of $f$ along the $i$ th unit direction $e^{i}$. If all the partial derivatives of $f$ exist at $x \in \mathbb{R}^{d}$, then we may define the gradient of $f$ at $x$, given by

$$
\nabla f(x)=\left(\frac{\partial f}{\partial x_{1}}(x), \ldots, \frac{\partial f}{\partial x_{d}}(x)\right)^{\top}
$$

If a function is Lipschitz continuous, then it is continuous and differentiable almost everywhere. However, there is a function that is Lipschitz continuous but not differentiable. For example, $f(x)=|x|$ for $x \in \mathbb{R}$. In addition, $f(x)=\|x\|_{1}$ for $x \in \mathbb{R}^{d}$.
Next, we consider

$$
\frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}(x) \quad \text { for } i, j \in[d]
$$

are the second partial derivatives of $f$. If all the second partial derivatives exist, then we may define the Hessian of $f$ as follows

$$
\nabla^{2} f(x)=\left(\begin{array}{cccc}
\frac{\partial^{2} f}{\partial x_{1}^{2}}(x) & \frac{\partial^{2} f}{\partial x_{1} \partial x_{2}}(x) & \cdots & \frac{\partial^{2} f}{\partial x_{1} \partial x_{d}}(x) \\
\frac{\partial^{2} f}{\partial x_{2} \partial x_{1}}(x) & \frac{\partial^{2} f}{\partial x_{2}^{2}}(x) & \cdots & \frac{\partial^{2} f}{\partial x_{2} \partial x_{d}}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^{2} f}{\partial x_{d} \partial x_{1}}(x) & \frac{\partial^{2} f}{\partial x_{d} \partial x_{2}}(x) & \cdots & \frac{\partial^{2} f}{\partial x_{d}^{2}}(x)
\end{array}\right) .
$$

Moreover, if the second partial derivatives are continuous, then Schwarz's theorem implies that the Hessian is symmetric, i.e.,

$$
\frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}(x)=\frac{\partial^{2} f}{\partial x_{j} \partial x_{i}}(x) \quad \text { for every } i, j \in[d] .
$$

## 3 A little bit of matrix calculus

Let $A$ be an $n \times d$ matrix and $b \in \mathbb{R}^{n}$. Let $f(x)=g(A x-b)$. Then by the chain rule,

$$
\nabla f(x)=A^{\top} \nabla g(A x-b), \quad \nabla^{2} f(x)=A^{\top} \nabla^{2} g(A x-b) A .
$$

Example 2.1. Consider $f(x)=A x-b$. Then $\nabla f(x)=A^{\top}$.
Consider a quadratic function

$$
f(x)=x^{\top} Q x+p^{\top} x=\sum_{i=1}^{d} \sum_{j=1}^{d} Q_{i j} x_{i} x_{j}+\sum_{i=1}^{d} p_{i} x_{i} .
$$

Then the gradient of $f$ is given by

$$
\nabla f(x)=\left(Q+Q^{\top}\right) x+p, \quad \nabla^{2} f(x)=Q+Q^{\top} .
$$

Example 2.2. Consider $f(x)=\|A x-b\|_{2}^{2}$. Then

$$
\nabla f(x)=2 A^{\top}(A x-b), \quad \nabla^{2} f(x)=2 A^{\top} A
$$

## 4 Convex sets

### 4.1 Definition

Definition 2.3. A set $X \subseteq \mathbb{R}^{d}$ is convex if for any $u, v \in X$ and any $\lambda \in[0,1]$,

$$
\lambda u+(1-\lambda) v \in X
$$

In words, the line segment joining any two points is entirely contained the set. In Figure 2.1, we have a convex set and a non-convex set.


Figure 2.1: A convex set and a nonconvex set

Definition 2.4. Given $v^{1}, \ldots, v^{k} \in \mathbb{R}^{d}$, any linear combination $\lambda_{1} v^{1}+\cdots+\lambda_{k} v^{k}$ is a convex combination of $v^{1}, \ldots, v^{k}$ if

$$
\sum_{i=1}^{k} \lambda_{i}=1 \quad \text { and } \quad \lambda_{i} \geq 0 \text { for } i=1, \ldots, k
$$

The convex combination of two distinct points $u, v$ is the line segment $\{\lambda u+(1-\lambda) v: 0 \leq \lambda \leq 1\}$ connecting them.

Definition 2.5. The convex hull of a set $X$, denoted $\operatorname{conv}(X)$, is the set of all convex combinations of points in $X$. By definition,

$$
\operatorname{conv}(X)=\left\{\begin{array}{c}
n \in \mathbb{N}, v^{1}, \ldots, v^{n} \in X \\
\sum_{i=1}^{n} \lambda_{i} v^{i}: \quad \sum_{i=1}^{n} \lambda_{i}=1, \lambda_{1}, \ldots, \lambda_{n} \geq 0
\end{array}\right\}
$$

Here, $\operatorname{conv}(X)$ is always convex regardless of $X$. Figure 2.2 shows some examples of taking the convex hull of a set.


Figure 2.2: A convex set and a nonconvex set

### 4.2 Cones and affine subspaces

Definition 2.6. A set $C \subseteq \mathbb{R}^{d}$ is a cone if for any $v \in C$ and $\alpha>0$, we have $\alpha v \in C$. Furthermore, if $C$ is convex, then it is called a convex cone.

Note that not all cones are convex.
Definition 2.7. Given $v^{1}, \ldots, v^{k} \in \mathbb{R}^{d}$, any point of the form $\alpha_{1} v^{1}+\cdots+\alpha v_{k} v^{k}$ is a conic combination of $v^{1}, \ldots, v^{k}$ if $\alpha_{1}, \ldots, \alpha_{k} \geq 0$.

In other words, any nonnegative linear combination is a conic combination.
Definition 2.8. The conic hull of a set $X$, denoted cone $(X)$, is the set of all conic combinations of points in $X$. By definition,

$$
\operatorname{conv}(X)=\left\{\begin{array}{l}
\left.\sum_{i=1}^{n} \lambda_{i} v^{i}: \begin{array}{l}
n \in \mathbb{N}, v^{1}, \ldots, v^{n} \in X, \\
\lambda_{1}, \ldots, \lambda_{n} \geq 0
\end{array}\right\} . . . . ~ . ~
\end{array}\right.
$$

As $\operatorname{conv}(X), \operatorname{cone}(X)$ is always convex, Figure 2.3 shows an example taking the conic hull of a set in $\mathbb{R}^{2}$.
Lastly, we define the notion of affine subspaces.
Definition 2.9. Given $v^{1}, \ldots, v^{k} \in \mathbb{R}^{d}$, any point of the form $\theta_{1} v^{1}+\cdots+\theta v_{k} v^{k}$ is a affine combination of $v^{1}, \ldots, v^{k}$ if $\theta_{1}+\cdots+\theta_{k}=1$.


Figure 2.3: Taking the conic hull of a triangle in $\mathbb{R}^{2}$

In contrast to covex combinations, affine combinations allow negative multipliers.
Definition 2.10. The affine hull of a set $X$ is the set of all affine combinations of points in $X$.
The affine hull of $X$ is also referred to as the affine subspace spanned by $X$. In the previous lecture, we defined the linear subspace spanned by a finite set of vectors, but we can extend the definition to an arbitrary set. The linear hull of a set $X$ is equivalent to the linear subspace spanned by $X$.

In Figure 2.4, we have a set $S$ of two points in $\mathbb{R}^{2}$. The red line segment is $\operatorname{conv}(S)$, the green line through the two points is the affine subspace spanned by $S$, the blue cone depicts cone $(S)$, and lastly, the orange region (in fact, $\mathbb{R}^{2}$ ) is the linear subspace spanned by $S$.


Figure 2.4: Comparing the linear subspace, the affine subspace, the convex hull, and the conic hull

Theorem 2.11. An affine subspace is a translation of a linear subspace. For an affine subspace $V \subseteq \mathbb{R}^{d}$, there exist matrices $A$ and $b$ such that $V=\left\{x \in \mathbb{R}^{d}: A x=b\right\}$.

### 4.3 Examples

We saw that the convex hull and conic hull of a set are convex and that the linear subspace and affine subspace spanned by a set are convex. There are many more examples.

1. Empty set, singletons (sets of the form $\{v\}$ ),
2. Norm ball: $\left\{x \in \mathbb{R}^{d}:\|x-c\| \leq r\right\}$ where $c$ is the center.
3. Ellipsoid: $\left\{x \in \mathbb{R}^{d}:(x-c)^{\top} P(x-c) \leq 1\right\}$ where $P$ is positive definitie and $c$ is the center.
4. Hyperplane: $\left\{x \in \mathbb{R}^{d}: a^{\top} x=b\right\}$ where $a \in \mathbb{R}^{d}$ and $b \in \mathbb{R}$.
5. Half-space: $\left\{x \in \mathbb{R}^{d}: a^{\top} x \leq b\right\}$ where $a \in \mathbb{R}^{d}$ and $b \in \mathbb{R}$.
6. Polyhedron: A polyhedron is a finite intersection of half spaces, $\left\{x \in \mathbb{R}^{d}: A x \leq b\right\}$ where $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^{m}$. Here, $A x \leq b$ is a short-hand notation for system $a_{k}^{\top} x \leq b_{k}$ for $k \in[m]$.
7. Polytope: A polytope is a polyhedron that is bounded. Equivalently, a polytope is the convex hull of some finite set of vectors.
8. Simplex: A set of the form $\left\{x \in \mathbb{R}^{d}: 1^{\top} x=1, x \geq 0\right\}$, which is equal to the convex hull of $e^{1}, \ldots, e^{d}$, the $d$-dimensional unit vectors.
9. Nonnegative orthant: $\mathbb{R}_{+}^{d}=\left\{x \in \mathbb{R}^{d}: x \geq 0\right\}$.
10. Positive orthant: $\mathbb{R}_{++}^{d}=\left\{x \in \mathbb{R}^{d}: x>0\right\}$.

There are examples of convex cones, which are convex as well.

1. Norm cone: $\left\{(x, t) \in \mathbb{R}^{d} \times \mathbb{R}:\|x\| \leq t\right\}$. When $\|\cdot\|$ is the Euclidean norm, the cone is called the second-order cone.
2. Positive semidefinite cone: The set of all positive semidefinite matrices of a fixed dimension.

## 5 Convex functions

### 5.1 Definition

Definition 2.12. A function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is convex if the domain, denoted $\operatorname{dom}(f)$, is convex and for all $x, y \in \operatorname{dom}(f)$, we have

$$
f(\lambda x+(1-\lambda) y) \leq \lambda f(x)+(1-\lambda) f(y) \quad \text { for } 0 \leq \lambda \leq 1 .
$$

In words, function $f$ evaluated at a point between $x$ and $y$ lies below the line segment joining $f(x)$ and $f(y)$.


Figure 2.5: Illustration of a convex function in $\mathbb{R}^{2}$

Definition 2.13. We say that $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is concave if $-f$ is convex.
Definition 2.14. A function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is

- strictly convex if $\operatorname{dom}(f)$ is convex and for any distinct $x, y \in \operatorname{dom}(f)$, we have

$$
f(\lambda x+(1-\lambda) y)<\lambda f(x)+(1-\lambda) f(y) \text { for } 0<\lambda<1 .
$$

- strongly convex if $f(x)-\alpha\|x\|^{2}$ is convex for some $\alpha>0$ and norm $\|\cdot\|$.

Note that strong convexity implies strict convexity, and strict convexity implies convexity.

