IE 539 Convex Optimization Final Exam

Fall 2023

Out: 11st December 2023

Due: 16th December 2023 at 11:59pm

Instructions

- Submit a PDF document with your solutions through the assignment portal on KLMS by the due date. Please ensure that your name and student ID are on the front page.
- Late submissions will **not** be accepted except in extenuating circumstances. Special consideration should be applied for in this case.
- It is **required** that you typeset your solutions in LaTeX. Handwritten solutions will not be accepted.
- Spend some time ensuring your arguments are **coherent** and your solutions **clearly** communicate your ideas.

Question:	1	2	Total
Points:	40	60	100

1. We consider the following composite convex minimization problem

$$\text{minimize} \quad f(x) + g(x) \tag{1}$$

where f and g are closed and convex functions whose associated prox operators $\text{prox}_f(\cdot)$ and $\text{prox}_g(\cdot)$ are known. Let us consider

$$H(y) = y + \text{prox}_g\left(2\text{prox}_f(y) - y\right) - \text{prox}_f(y).$$

(a) (20 points) Prove that if $H(y) = y$, then $x = \text{prox}_f(y)$ is an optimal solution to problem (1).

(b) (20 points) Part (a) suggests an algorithm for solving (1) as follows. Given (x_t, y_t), we deduce (x_{t+1}, y_{t+1}) as

$$\begin{aligned} y_{t+1} &= y_t + \text{prox}_g(2x_t - y_t) - x_t \\ x_{t+1} &= \text{prox}_f(y_{t+1}). \end{aligned} \tag{2}$$

Prove that the algorithm with the update rule (2) is equivalent to the algorithm presented next. Given (x_t, u_t, w_t), we obtain $(x_{t+1}, u_{t+1}, w_{t+1})$ as

$$\begin{aligned} u_{t+1} &= \text{prox}_g(x_t + w_t) \\ x_{t+1} &= \text{prox}_f(u_{t+1} - w_t) \\ w_{t+1} &= w_t + x_{t+1} - u_{t+1} \end{aligned} \tag{3}$$

2. We consider the following composite convex minimization problem

$$\begin{aligned} \text{minimize} \quad & f_1(x_1) + f_2(x_2) \\ \text{subject to} \quad & A_1 x_1 + A_2 x_2 = b. \end{aligned} \tag{4}$$

We learned that the dual of (4) is given by (or equivalent to)

$$\text{minimize} \quad b^\top z + f_1^*\left(-A_1^\top z\right) + f_2^*\left(-A_2^\top z\right). \tag{5}$$

To solve (5), we consider an algorithm that proceeds with the following update rule. Given (z_t, u_t, w_t), we obtain $(z_{t+1}, u_{t+1}, w_{t+1})$ as

$$\begin{aligned} u_{t+1} &= \text{prox}_{\eta g}(z_t + w_t) \\ z_{t+1} &= \text{prox}_{\eta f}(u_{t+1} - w_t) \\ w_{t+1} &= w_t + z_{t+1} - u_{t+1} \end{aligned} \tag{6}$$

where

$$f(z) = f_2^*\left(-A_2^\top z\right) \quad \text{and} \quad g(z) = b^\top z + f_1^*\left(-A_1^\top z\right).$$

Here, z_t is a solution to (5) while u_t and w_t are auxiliary variables.

(a) (20 points) Prove that u_{t+1} is given by

$$u_{t+1} = z_t + w_t + \eta\left(A_1 x_{1,t} - b\right)$$

where

$$x_{1,t} = \arg\min_{x_1} \left\{ f_1(x_1) + (z_t + w_t)^\top \left(A_1 x_1 - b\right) + \frac{\eta}{2}\|A_1 x_1 - b\|_2^2 \right\}.$$

(b) (20 points) Prove that z_{t+1} is given by

$$z_{t+1} = z_t + \eta\left(A_1 x_{1,t} + A_2 x_{2,t} - b\right)$$

where

$$x_{2,t} = \arg\min_{x_2} \left\{ f_2(x_2) + z_t^\top A_2 x_2 + \frac{\eta}{2}\|A_1 x_{1,t} + A_2 x_2 - b\|_2^2 \right\}.$$

(c) (20 points) Using parts (a) and (b), prove that (6) is equivalent to

$$z_{t+1} = z_t + \eta\left(A_1 x_{1,t} + A_2 x_{2,t} - b\right)$$

where

$$\begin{aligned} x_{1,t} &= \arg\min_{x_1} \left\{ f_1(x_1) + z_t^\top A_1 x_1 + \frac{\eta}{2}\|A_1 x_1 + A_2 x_{2,t-1} - b\|_2^2 \right\}, \\ x_{2,t} &= \arg\min_{x_2} \left\{ f_2(x_2) + z_t^\top A_2 x_2 + \frac{\eta}{2}\|A_1 x_{1,t} + A_2 x_2 - b\|_2^2 \right\}. \end{aligned}$$