IE 539 Convex Optimization Assignment 1

Fall 2023

Out: 12th September 2023

Due: 25th September 2023 at 11:59pm

Instructions

- Submit a PDF document with your solutions through the assignment portal on KLMS by the due date. Please ensure that your name and student ID are on the front page.
- **Late assignments will not be accepted** except in extenuating circumstances. Special consideration should be applied for in this case.
- It is **required** that you **typeset your solutions in LaTeX**. Handwritten solutions will not be accepted.
- Spend some time ensuring your arguments are **coherent** and your solutions **clearly** communicate your ideas.

Question:	1	2	3	4	Total
Points:	10	20	35	35	100

1. (10 points) Prove that a function is covex if and only if its epigraph is a convex set.

2. (20 points) Suppose that $g_1, \dots, g_p : \mathbb{R}^d \to \mathbb{R}$ are convex and $h_1, \dots, h_q : \mathbb{R}^d \to \mathbb{R}$ are affine functions. Prove that
$$C := \{x \in \mathbb{R}^d : \; g_i(x) \le 0 \text{ for } i = 1, \dots, p, \; h_i(x) = 0 \text{ for } j = 1, \dots, q\}$$
is a convex set.

3. Verify convexity/concavity of the following functions. You may use the first-order and second-order characterizations of convex functions, while there exists a direct proof based on the definition of convex functions.

   (a) (5 points) The *negative entropy function* is convex on $\mathbb{R}^d_{++}$:
   $$f(x) := \sum_{i \in [d]} x_i \log(x_i).$$

   (b) (10 points) The geometric mean is concave on $\mathbb{R}^d_{++}$:
   $$f(x) = \left(\prod_{i \in [d]} x_i \right)^{1/d}.$$
   [Hint: compute $\frac{\lambda f(x) + (1-\lambda) f(y)}{f(\lambda x + (1-\lambda) y)}$. You may use (without proof) the arithmetic-geometric mean inequality: $\left(\prod_{i \in [d]} x_i\right)^{1/d} \le \frac{1}{d} \sum_{i \in [d]} x_i$.]

   (c) (10 points) The *conjugate* of a function $f : \mathbb{R}^d \to \mathbb{R}$:
   $$f^*(x) = \sup_{y \in \mathbb{R}^d} \{\langle y, x \rangle - f(y)\}$$

   (d) (10 points) The sum of k largest components of $x \in \mathbb{R}^d$:
   $$f(x) = x_{\sigma(1)} + \cdots + x_{\sigma(k)}$$
   where $x_{\sigma(1)} \ge \cdots \ge x_{\sigma(d)}$ are the rearrangement of $x_1, \dots, x_d$ in nonincreasing order.

4. This question asks you to show that the following formulation for uncertainty quantification is a convex optimization problem.
$$\begin{aligned} \text{maximize} \quad & x^\top (\bar{\Sigma} + S) x \\ \text{subject to} \quad & \bar{\Sigma} + S \succeq 0, \\ & \|S\|_{\text{nuc}} \le \epsilon, \\ & S \in \mathbb{R}^{d \times d} \end{aligned}$$
where $\bar{\Sigma}$ is an empirical covariance matrix and $A \succeq 0$ means matrix A being positive semidefinite.

   (a) (20 points) The *nuclear nor*m of a square matrix $S \in \mathbb{R}^{d \times d}$ is defined as
   $$\|S\|_{\text{nuc}} = \sum_{i=1}^{d} \sqrt{\lambda_i(S^\top S)}$$
   where $\lambda_1(S^\top S), \dots, \lambda_d(S^\top S)$ are the eigenvalues of $S^\top S$. Prove that the nuclear norm is a norm.

   (b) (10 points) Prove that
   $$\left\{ S \in \mathbb{R}^{d \times d} : \; \bar{\Sigma} + S \succeq 0 \right\}$$
   is a convex set.

   (c) (5 points) Prove that the formulation is a convex optimization problem.