1 Outline

In this lecture, we cover

- deriving the dual linear program,
- linear programming duality theorems.

2 Deriving the dual linear program

Suppose that we have a linear program in the most general form as follows.

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \leq b_i, \quad i \in M_L \\ & a_i^\top x = b_i, \quad i \in M_E \\ & a_i^\top x \geq b_i, \quad i \in M_G \\ & x_j \leq 0, \quad j \in D_L \\ & x_j \text{ free}, \quad j \in D_F \\ & x_j \geq 0, \quad j \in D_G \end{aligned}$$

where $M_L, M_E, M_G \subseteq [m]$ and $D_L, D_F, D_G \subseteq [d]$ are index partitions. To provide a strong lower bound on this linear program, we derive the **dual linear program**. As in the example from the last lecture, we take multipliers for the constraints and aggregate them to provide a lower bound on the objective function $c^\top x$. We call this procedure **constraint aggregation**. From this process, we obtain the dual linear program where the multipliers are used as variables. Hence, we refer to the multipliers as **dual variables**. Let us outline the procedure of deriving the dual LP with the above general LP.

Step 1: assign dual variables to each constraint, except sign constraints. We assign a dual variable/multiplier to each constraint.

$$\begin{aligned} a_i^\top x \leq b_i \quad & (\boldsymbol{\lambda_i \leq 0}), \quad i \in M_L \\ a_i^\top x = b_i \quad & (\boldsymbol{\lambda_i} \textbf{ free}), \quad i \in M_E \\ a_i^\top x \geq b_i \quad & (\boldsymbol{\lambda_i \geq 0}), \quad i \in M_G \end{aligned}$$

Here, the signs of dual variables should be the same as the corresponding inequality directions. In the previous example

$$\begin{aligned} \min \quad & 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & 2x_1 + 4x_2 = 1 \\ & 3x_1 - x_2 + x_3 = 4 \\ & x_1, x_2, x_3 \geq 0, \end{aligned}$$

we have only equality constraints, and therefore, the corresponding dual variables are free variables. The purpose of imposing the signs this way is to provide a lower bound on the primal objective when we aggregate the constraints.

Step 2: aggregate the constraints via the dual variables. We first multiply each constraint by the corresponding dual variable.

$$\begin{aligned}\lambda_i a_i^\top x &\geq b_i\lambda_i, \quad i\in M_L\\ \lambda_i a_i^\top x &= b_i\lambda_i, \quad i\in M_E\\ \lambda_i a_i^\top x &\geq b_i\lambda_i, \quad i\in M_G\end{aligned}$$

By the proper choice of the signs of dual variables, the resulting inequalities after multiplying constraints by dual variables all have the same sign.

Then we sum up the resulting inequalities/equalities.

$$\begin{aligned}\sum_{i\in M_L} b_i\lambda_i + \sum_{i\in M_E} b_i\lambda_i + \sum_{i\in M_G} b_i\lambda_i &\leq \sum_{i\in M_L}\lambda_i a_i^\top x + \sum_{i\in M_E}\lambda_i a_i^\top x + \sum_{i\in M_G}\lambda_i a_i^\top x\\ &= \left(\sum_{i\in M_L}\lambda_i a_i + \sum_{i\in M_E}\lambda_i a_i + \sum_{i\in M_G}\lambda_i a_i\right)^\top x.\end{aligned}$$

Let b be the right-hand side vector that consists of b_i for $i\in M_L\cup M_E\cup M_G$, and let λ be the corresponding vector of dual variables. Then

$$\sum_{i\in M_L} b_i\lambda_i + \sum_{i\in M_E} b_i\lambda_i + \sum_{i\in M_G} b_i\lambda_i = b^\top\lambda.$$

Moreover, let A be the constraint matrix whose rows are $a_i^\top$ for $i\in M_L\cup M_E\cup M_G$. Then

$$\sum_{i\in M_L}\lambda_i a_i + \sum_{i\in M_E}\lambda_i a_i + \sum_{i\in M_G}\lambda_i a_i = A^\top\lambda.$$

Therefore, the resulting inequality is equivalent to

$$b^\top\lambda \leq (A^\top\lambda)^\top x = \lambda^\top Ax.$$

Step 3: match with the primal objective vector. We have $b^\top\lambda\leq(A^\top\lambda)^\top x$. Here, the right-hand side is a linear function in x, and we want to use it to lower bound the primal objective $c^\top x$. Basically, we want that

$$(A^\top\lambda)^\top x \leq c^\top x.$$

Note that the jth component of $A^\top\lambda$ is $\tilde{a}_j^\top\lambda$ where $\tilde{a}_j$ is the jth column of A. Then

$$(A^\top\lambda)^\top x = \sum_{j\in[d]}(\tilde{a}_j^\top\lambda)x_j = \sum_{j\in D_L}(\tilde{a}_j^\top\lambda)x_j + \sum_{j\in D_F}(\tilde{a}_j^\top\lambda)x_j + \sum_{j\in D_G}(\tilde{a}_j^\top\lambda)x_j.$$

Recall that $x_j\leq 0$ for $j\in D_L$, x_j is free for $j\in D_F$, and $x_j\geq 0$ for $j\in D_G$. Hence, if

$$\begin{aligned}\tilde{a}_j^\top\lambda &\geq c_j, \quad j\in D_L\\ \tilde{a}_j^\top\lambda &= c_j, \quad j\in D_F\\ \tilde{a}_j^\top\lambda &\leq c_j, \quad j\in D_G,\end{aligned}$$

then it follows that

$$\begin{aligned}(A^\top \lambda)^\top x &= \sum_{j\in D_L} (\tilde{a}_j^\top \lambda)x_j + \sum_{j\in D_F} (\tilde{a}_j^\top \lambda)x_j + \sum_{j\in D_G} (\tilde{a}_j^\top \lambda)x_j. \\ &\leq \sum_{j\in D_L} c_j x_j + \sum_{j\in D_F} c_j x_j + \sum_{j\in D_G} c_j x_j \\ &= c^\top x.\end{aligned}$$

Step 4: obtain the dual linear program. To summarize what we derived, we obtained $b^\top \lambda \leq (A^\top \lambda)^\top x \leq c^\top x$ when λ satisfies certain conditions. Here, $b^\top \lambda$ would be a lower bound on $c^\top x$. Then by maximizing the value of $b^\top \lambda$ over all λ's satisfying the conditions, we derive the best possible lower bound with this procedure. Then we obtain the dual linear program as follows.

$$\begin{aligned}\max\quad & \sum_{i\in M_L} b_i\lambda_i + \sum_{i\in M_E} b_i\lambda_i + \sum_{i\in M_G} b_i\lambda_i \\ \text{s.t.}\quad & \lambda_i \leq 0, \quad i \in M_L \\ & \lambda_i \text{ free}, \quad i \in M_E \\ & \lambda_i \geq 0, \quad i \in M_G \\ & \tilde{a}_j^\top \lambda \geq c_j, \quad j \in D_L \\ & \tilde{a}_j^\top \lambda = c_j, \quad j \in D_F \\ & \tilde{a}_j^\top \lambda \leq c_j, \quad j \in D_G.\end{aligned}$$

Let us compare the dual linear program and the primal linear program. Note that a constraint of the primal corresponds to a variable in the dual. A variable in the primal corresponds to a constraint in the dual.

Example 9.1. The dual of

$$\begin{aligned}\min\quad & 2x_1 + 3x_2 \\ \text{s.t.}\quad & 7x_1 + 4x_2 \leq 1 \\ & 5x_1 + 9x_2 = 3 \\ & x_1 \geq 0, x_2 \leq 0\end{aligned}$$

is given by

$$\begin{aligned}\max\quad & \lambda_1 + 3\lambda_2 \\ \text{s.t.}\quad & 7\lambda_1 + 5\lambda_2 \leq 2 \\ & 4\lambda_1 + 9\lambda_2 \geq 3 \\ & \lambda_1 \leq 0, \lambda_2 \text{ free.}\end{aligned}$$

Example 9.2. The dual of

$$\begin{aligned}\min\quad & 2x_1 + 3x_2 \\ \text{s.t.}\quad & 7x_1 + 4x_2 \geq 1 \\ & 5x_1 + 9x_2 \leq 3 \\ & x_1 \leq 0, x_2 \leq 0\end{aligned}$$

is given by

$$\begin{aligned} \max \quad & \lambda_1 + 3\lambda_2 \\ \text{s.t.} \quad & 7\lambda_1 + 5\lambda_2 \geq 2 \\ & 4\lambda_1 + 9\lambda_2 \geq 3 \\ & \lambda_1 \geq 0, \lambda_2 \leq 0. \end{aligned}$$

What if the primal LP is a maximization problem? There are two options.

1. We turn the LP into a minimization problem. When the objective is to maximize $c^\top x$, it is equivalent to minimize $-c^\top x$. Then we use the result for deriving the dual LP of a minimizing linear program.
2. We may repeat the process of deriving the dual LP of a maximizing linear program.

Example 9.3. Let us take the first approach to derive the dual of

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & 7x_1 + 4x_2 \leq 1 \\ & 5x_1 + 9x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

This LP is equivalent to

$$(-1) \quad \times \quad \begin{aligned} \min \quad & -2x_1 - 3x_2 \\ \text{s.t.} \quad & 7x_1 + 4x_2 \leq 1 \\ & 5x_1 + 9x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

The dual of the minimizing program is given by

$$\begin{aligned} \max \quad & \lambda_1 + 3\lambda_2 \\ \text{s.t.} \quad & 7\lambda_1 + 5\lambda_2 \leq -2 \\ & 4\lambda_1 + 7\lambda_2 \leq -3 \\ & \lambda_1 \leq 0, \lambda_2 \leq 0. \end{aligned}$$

Then the dual LP of the first linear program is given by multiplying this LP by (-1):

$$\begin{aligned} \min \quad & -\lambda_1 - 3\lambda_2 \\ \text{s.t.} \quad & 7\lambda_1 + 5\lambda_2 \leq -2 \\ & 4\lambda_1 + 7\lambda_2 \leq -3 \\ & \lambda_1 \leq 0, \lambda_2 \leq 0. \end{aligned}$$

Replacing λ_1 by $-\lambda_1$ and λ_2 by $-\lambda_2$, we get

$$\begin{aligned} \min \quad & \lambda_1 + 3\lambda_2 \\ \text{s.t.} \quad & 7\lambda_1 + 5\lambda_2 \geq 2 \\ & 4\lambda_1 + 7\lambda_2 \geq 3 \\ & \lambda_1 \geq 0, \lambda_2 \geq 0. \end{aligned}$$

3 Deriving the dual linear program in matrix form

Let us consider the following linear program

$$\begin{aligned} p^* \quad = \quad \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b. \end{aligned}$$

Let us derive the dual linear program of this. Although the constraints are written in matrix form, we apply the same approach.

1. In this linear program, we take $\lambda \in \mathbb{R}^m$ with $\lambda \geq 0$ where m is the number of rows in A.
2. Multiplying the system by λ, it follows that
$$b^\top \lambda \leq \lambda^\top Ax = (A^\top \lambda)^\top x.$$
3. Note that x variables are free variables. Then we impose that
$$c = A^\top \lambda.$$
   This implies that
$$c^\top x = (A^\top \lambda)^\top x \geq b^\top \lambda.$$
4. Then the dual linear program is given by
$$\begin{aligned} d^* \quad = \quad \max \quad & b^\top \lambda \\ \text{s.t.} \quad & A^\top \lambda = c \\ & \lambda \geq 0 \end{aligned}$$

Let us consider the following linear program in standard form.

$$\begin{aligned} p^* \quad = \quad \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Let us derive the dual linear program of this.

1. We take free dual variables $\lambda \in \mathbb{R}^m$ where m is the number of rows in A.
2. Multiplying the system by λ, it follows that
$$b^\top \lambda = \lambda^\top Ax = (A^\top \lambda)^\top x.$$
3. Note that $x \geq 0$, so we impose that
$$c \geq A^\top \lambda.$$
   This implies that
$$c^\top x \geq (A^\top \lambda)^\top x = b^\top \lambda.$$
4. Then the dual linear program is given by
$$\begin{aligned} d^* \quad = \quad \max \quad & b^\top \lambda \\ \text{s.t.} \quad & A^\top \lambda \leq c. \end{aligned}$$

4 Linear programming duality

Let the primal linear program is a minimizing program. Then the dual linear program is a maximizing program.

Theorem 9.4 (Weak duality)**.** *Let p^* and d^* be the primal and dual optimal values. Then*

$$p^* \geq d^*.$$

Consequently, the following statements hold.

- If the primal LP is unbounded, then $p^* = d^* = -\infty$, and therefore, the dual LP is infeasible.
- If the dual LP is unbounded, then $p^* = d^* = \infty$, and therefore, the primal LP is infeasible.

Moreover, for any x feasible to the primal LP and any λ feasible to the dual LP, we have

$$c^\top x \geq b^\top \lambda.$$

In fact, it is possible that both primal and dual are infeasible.

Example 9.5. Note that the linear program

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & x_1 + x_2 = 2 \end{aligned}$$

has its dual LP given by

$$\begin{aligned} \max \quad & \lambda_1 + 2\lambda_2 \\ \text{s.t.} \quad & \lambda_1 + \lambda_2 = 1 \\ & \lambda_1 + \lambda_2 = 2. \end{aligned}$$

In fact, these two linear programs are identical and infeasible.

Theorem 9.6 (Strong duality)**.** *Let p^* and d^* be the primal and dual optimal values. If any of*

- *both primal and dual are feasible,*
- *primal has a finite optimal value,*
- *dual has a finite optimal value*

holds, then we have

$$p^* = d^*.$$