1 Outline

In this lecture, we cover

- simplex method technical details,
- upper and lower bounds on a linear program.

2 Simplex method phase I technical details

Recall that the two-phase simplex method starts with the step of finding a feasible dictionary. This step is essentially checking the feasibility of the linear program.

2.1 Inequality constraints

Assume that the constraints are given by

$$Ax \leq b, \quad x \geq 0.$$

Then adding slack variables s, we obtain an equivalent system

$$\begin{aligned} &Ax + s = \begin{bmatrix} a_1^\top x + s_1 \\ \vdots \\ a_m^\top x + s_m \end{bmatrix} = b \\ &x \geq 0, s \geq 0. \end{aligned}$$

This gives rise to the initial dictionary

$$\underbrace{\begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}}_{s} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{b} + \underbrace{\begin{bmatrix} -a_1^\top x \\ \vdots \\ -a_m^\top x \end{bmatrix}}_{-Ax}.$$

If the components of b are all nonnegative, then it is a feasible dictionary. If not,

$$b_{\min} = \min_{i \in [m]} b_i$$

is negative. In this case, we consider

$$Ax - t\mathbf{1} \leq b, \quad x \geq 0, t \geq 0$$

where $\mathbf{1}$ the vector of all ones. In conrast to the original system, this new system with variable t is always feasible. In fact,

$$(x, t) = (0, -b_{\min})$$

is a feasible solution. Moreover, consider

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & Ax - t\mathbf{1} \le b \\ & x \ge 0, t \ge 0 \end{aligned}$$

which we refer to as the **Phase I LP**. We know that system $Ax \le b, x \ge 0$ is feasible if and only if the Phase I LP has optimal value 0. That is because the optimal value being 0 means that there exists a solution $(x,t) = (\bar{x}, 0)$ that satisfies $A\bar{x} - 0\mathbf{1} = A\bar{x} \le b$ and $\bar{x} \ge 0$.

To solve the Phase I LP, we obtain its standard form, given by

$$\begin{aligned} & Ax - t\mathbf{1} + s = \begin{bmatrix} a_1^\top x - t + s_1 \\ \vdots \\ a_m^\top x - t + s_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \\ & x \ge 0, t \ge 0, s \ge 0. \end{aligned}$$

Let us get the initial dictionary of this. For simpler presentation, let us assume that b_1 is the smallest among the components of b. Subtracting the first row from the other rows, we obtain

$$\begin{bmatrix} a_1^\top x - t + s_1 \\ (a_2 - a_1)^\top x + s_2 - s_1 \\ \vdots \\ (a_m - a_1)^\top x + s_m - s_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ \vdots \\ b_m - b_1 \end{bmatrix}.$$

Moving the variables accordingly, we get

$$\begin{bmatrix} t \\ s_2 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} -b_1 \\ b_2 - b_1 \\ \vdots \\ b_m - b_1 \end{bmatrix} + \begin{bmatrix} a_1^\top x + s_1 \\ -(a_2 - a_1)^\top x + s_1 \\ \vdots \\ -(a_m - a_1)^\top x + s_1 \end{bmatrix}.$$

Here, this dictionary is feasible because

$$b_i - b_1 = b_i - \min\{b_j : \ j \in [m]\} \ge 0$$

for all $i \in [m]$. Then we may proceed the simplex algorithm to solve the Phase I LP.

2.2 Equality constraints

Now assume that the constraints of a linear program are given by

$$Ax = b, \quad x \ge 0.$$

In this case, what would be the right form for the Phase I LP? We may apply the same idea! We consider

$$\begin{aligned} \min \quad & t + \sum_{i=1}^m s_i \\ \text{s.t.} \quad & Ax - t\mathbf{1} + s = b \\ & x \ge 0, s \ge 0, t \ge 0. \end{aligned}$$

where $\mathbf{1}$ is the veector of all ones. Here s is the vector of m variables and t is a single variable. This is the **Phase I LP** for the equality constrained case.

If all components of b are nonnegative,

$$(x, t, s) = (0, 0, b)$$

is a feasible solution! Moreover,

$$\begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} + \begin{bmatrix} -a_1^\top x + t \\ \vdots \\ -a_m^\top x + t \end{bmatrix}$$

is a feasible dictionary.

Next, consider the case when b has some negative component. WIthout loss of generality, we may assume that b_1 has the smallest value among the components of b. In this case, we use t instead of s_1 for a basic variable. Then we obtain from $Ax - t\mathbf{1} + s = b$ that

$$\begin{bmatrix} t \\ s_2 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} -b_1 \\ b_2 - b_1 \\ \vdots \\ b_m - b_1 \end{bmatrix} + \begin{bmatrix} a_1^\top x + s_1 \\ -(a_2 - a_1)^\top x + s_1 \\ \vdots \\ -(a_m - a_1)^\top x + s_1 \end{bmatrix}.$$

This is a feasible dictionary.

Theorem 8.1. *The system $Ax = b$, $x \geq 0$ is feasible if and only if the Phase I LP has optimal value 0.*

Proof. The system $Ax = b$, $x \geq 0$ is feasible if and only if there exists (x, t, s) with $t = 0$ and $s = 0$ satisfies $Ax - t\mathbf{1} + s = b$ with $x \geq 0$, in which case (x, t, s) has objective value 0. □

3 Phase II technical details

In this section, we review the second phase of the simplex algorithm for general linear programs. Consider a linear program in standard form as follows.

$$\begin{aligned} \max \quad & z = c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Assume that we are given a feasible dictionary, which is guaranteed after the first phase of the simplex algorithm. Suppose that the vector of variables x has **basic variables** x_B and **non-basic variables** x_N, i.e., we may write x as

$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}.$$

Moreover, we decompose the objective coefficient vector c and the constraint matrix A with respect to the basic and non-basic variables. Basically,

$$c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}, \quad A = \begin{bmatrix} B & N \end{bmatrix}.$$

Assumption 1. To set x_B as basic variables, the corresponding constraint matrix B should be nonsingular, i.e., B^{-1} exists.

Then the linear program in standard form can be rewritten as

$$\begin{aligned} \max \quad & z = c_B^\top x_B + c_N^\top x_N \\ \text{s.t.} \quad & Bx_B + Nx_N = b \\ & x_B \geq 0, x_N \geq 0. \end{aligned}$$

To obtain the feasible dictionary with respect to the basic variables x_B, we need to apply the required row operations. In fact, applying the required row operations is equivalent to multiplying both sides by B^{-1}. It follows from $Bx_B + Nx_N = b$ that

$$x_B = B^{-1}b - B^{-1}Nx_N.$$

Assumption 2. For the basic variables x_B to give a feasible dictionary, all components of $B^{-1}b$ are nonnegative.

Next we need to eliminate the basic variables x_B from the objective row. To do so, we use

$$c_B^\top x_B = c_B^\top B^{-1}b - c_B^\top B^{-1}Nx_N.$$

We plug in this to the objective row to eliminate x_B. Note that

$$\begin{aligned} z &= c_B^\top x_B + c_N^\top x_N \\ &= c_B^\top B^{-1}b - c_B^\top B^{-1}Nx_N + c_N^\top x_N \\ &= c_B^\top B^{-1}b + (c_N - N^\top (B^{-1})^\top c_B)^\top x_N. \end{aligned}$$

Consequently, the dictionary is given by

$$\begin{aligned} z &= c_B^\top B^{-1}b & &+(c_N - N^\top (B^{-1})^\top c_B)^\top x_N \\ x_B &= B^{-1}b & &-B^{-1}Nx_N \end{aligned}$$

- The corresponding solution is given by

$$(x_B, x_N) = (B^{-1}b, 0).$$

- The corresponding objective value is

$$z = c_B^\top B^{-1}b.$$

- We call the matrix B or the columns corresponding to the basic variables x_B **basis**.
- The objective coefficients $c_N - N^\top (B^{-1})^\top c_B$ are called the **reduced costs**.
- For maximization, the current dictionary is optimal if the reduced costs are all non-positive.

4 Upper and lower bounds for a linear program

Let us consider the following linear program with three variables.

$$\begin{aligned} p^* := \min \quad & 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & 2x_1 + 4x_2 = 1 \\ & 3x_1 - x_2 + x_3 = 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Let us derive upper and lower bounds on p^*.

4.1 Upper bound

As this is a minimization problem, we just find a feasible solution, and it objective value would be an upper bound on p^*. For example,

$$(x_1, x_2, x_3) = \left(0, \frac{1}{4}, \frac{17}{4}\right)$$

is a feasible solution whose objective value is

$$4 \times 0 + \frac{1}{4} + 3 \times \frac{17}{4} = 13.$$

Therefore, we deduce that

$$p^* \leq 13.$$

For any feasible solution $(x_1, x_2, x_3) = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$,

$$p^* \leq 4\bar{x}_1 + \bar{x}_2 + 3\bar{x}_3.$$

4.2 Lower bound

What about lower bounds? Let us multiply $2x_1 + 4x_2 = 1$ by λ_1 and $3x_1 - x_2 + x_3 = 4$ by λ_2. Note that

$$\begin{aligned} \lambda_1 + 4\lambda_2 &= \lambda_1(2x_1 + 4x_2) + \lambda_2(3x_1 - x_2 + x_3) \\ &= (2\lambda_1 + 3\lambda_2)x_1 + (4\lambda_1 - \lambda_2)x_2 + \lambda_2 x_3 \end{aligned}$$

If (x_1, x_2, x_3) is feasible, then x_1, x_2, x_3 are all nonnegative. What this means is that if λ_1 and λ_2 satisfy

$$\begin{aligned} 2\lambda_1 + 3\lambda_2 &\leq 4, \\ 4\lambda_1 - \lambda_2 &\leq 1, \\ \lambda_2 &\leq 3, \end{aligned}$$

then

$$\begin{aligned} (2\lambda_1 + 3\lambda_2)x_1 &\leq 4x_1, \\ (4\lambda_1 - \lambda_2)x_2 &\leq x_2, \\ \lambda_2 x_3 &\leq 3x_3. \end{aligned}$$

Summing these up, we obtain

$$\begin{aligned} 4x_1 + x_2 + 3x_3 &\geq (2\lambda_1 + 3\lambda_2)x_1 + (4\lambda_1 - \lambda_2)x_2 + \lambda_2 x_3 \\ &= \lambda_1 + 4\lambda_2. \end{aligned}$$

Therefore, $\lambda_1 + 4\lambda_2$ for any (λ_1, λ_2) satisfying

$$2\lambda_1 + 3\lambda_2 \leq 4, \quad 4\lambda_1 - \lambda_2 \leq 1, \quad \lambda_2 \leq 3$$

is a lower bound on p^*.

- Choosing $(\lambda_1, \lambda_2) = (1/4, 0)$, $\lambda_1 + 4\lambda_2 = 1/4$ is a lower bound.
- Choosing $(\lambda_1, \lambda_2) = (0, 4/3)$, $\lambda_1 + 4\lambda_2 = 3$ is a lower bound.

What is the best lower bound can we obtain from this procedure? Let us compute the maximum value of $\lambda_1 + 4\lambda_2$ over all (λ_1, λ_2) satisfying the inequalities!

$$\begin{aligned} d^* := \max \quad & \lambda_1 + 4\lambda_2 \\ \text{s.t.} \quad & 2\lambda_1 + 3\lambda_2 \leq 4, \\ & 4\lambda_1 - \lambda_2 \leq 1, \\ & \lambda_2 \leq 3 \end{aligned}$$

In fact, this is also a linear program. We call this the **dual linear program (LP)**. We refer to the original linear program as the **primal LP**. Solving the dual LP, we get an optimal solution

$$(\lambda_1, \lambda_2) = \left(-\frac{5}{2}, 3\right),$$

in which case

$$d^* = \frac{19}{2}.$$

This provides a lower bound

$$d^* = \frac{19}{2} \leq p^*.$$

It turns out that this lower bound is in fact the best possible bound we can get. Note that

$$(x_1, x_2, x_3) = \left(\frac{1}{2}, 0, \frac{5}{2}\right)$$

is a feasible solution to the primal LP, and its objective value is precisely $19/2$. This proves that the solution is optimal to the primal LP, and moreover,

$$d^* = \frac{19}{2} = p^*.$$