1 Outline

In this lecture, we cover

- the two-phase simplex algorithm,
- recognizing infeasible and unbounded linear programs.

2 Two-phase simplex algorithm

In the last lecture, we applied the simplex method to solve the following linear program with two variables.

$$\begin{aligned} \max \quad & z = 5x + 4y \\ \text{s.t.} \quad & 2x + 3y \le 150, \\ & 2x + y \le 70, \\ & x, y \ge 0. \end{aligned}$$

This linear program can be converted into standard form, given by

$$\begin{aligned} \max_{x,y} \quad & z = 5x + 4y \\ \text{s.t.} \quad & 2x + 3y + s_1 = 150, \\ & 2x + y + s_2 = 70, \\ & x, y, s_1, s_2 \ge 0. \end{aligned}$$

Recall that the slack variables s_1 and s_1 naturally give rise to the initial dictionary as follows.

$$\begin{array}{rclccc} z & = & & +5x & +4y, \\ s_1 & = & 150 & -2x & -3y, \\ s_2 & = & 70 & -2x & -y. \end{array}$$

Then we obtain the initial solution $(x, y, s_1, s_2) = (0, 0, 150, 70)$. We say that this is a **feasible dictionary**.

Note that using the slack variables for the initial dictionary was feasible because the right-hand side values, 150 and 70, are all nonnegative. What about the following linear program?

$$\begin{aligned} \max \quad & z = x + 2y \\ \text{s.t.} \quad & 2x + 3y \le 150, \\ & -x + y \le -25, \\ & x, y \ge 0. \end{aligned}$$

As before, we use slack variables s_1 and s_2 to transform the inequality constraints into equalities as follows.

$$\begin{aligned} \max \quad & z = x + 2y \\ \text{s.t.} \quad & 2x + 3y + s_1 = 150, \\ & -x + y + s_2 = -25, \\ & x, y \geq 0, \end{aligned}$$

and this gives rise to the following dictionary.

$$\begin{array}{rcrrr} z & = & & +x & +2y, \\ s_1 & = & 150 & -2x & -3y, \\ s_2 & = & -25 & +x & -y. \end{array}$$

Here, if we set $x = y = 0$, then (s_1, s_2) would be $(150, -25)$, violating the nonnegativity constraint on s_2. We say that this is an **infeasible dictionary**. We cannot proceed the simplex algorithm with an infeasible dictionary.

Motivated by this, the **two-phase** simplex algorithm proceeds in the phase of finding a feasible dictionary and the solution phase. In the first phase, we check the feasibility of the problem. If the given linear program is feasible, then the first phase ends with a feasible dictionary. If not, we conclude that the linear program is infeasible. The solution phase is the simplex algorithm outlined in the last lecture.

2.1 Phase I: find a feasible disctionary

For the first phase, we consider another linear program to check the feasibility of the original linear program. For our example, we consider

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & 2x + 3y - t \leq 150, \\ & -x + y - t \leq -25, \\ & x, y, t \geq 0. \end{aligned}$$

Theorem 7.1. *The linear program is feasible and its optimal value is equal to 0 if and only if the original linear program is feasible.*

Proof. We may take a sufficiently large number for t to make the constraints always satisfied. Moreover, the optimal value is 0 if and only if there is a solution (x, y, t) with $t = 0$ that satisfies the constraints. □

The standard form is given by

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & 2x + 3y - t + s_1 = 150, \\ & -x + y - t + s_2 = -25, \\ & x, y \geq 0, \end{aligned}$$

Then the corresponding initial dictionary is given by

$$\begin{array}{rcrrrr} z & = & & & & -t, \\ s_1 & = & 150 & -2x & -3y & +t, \\ s_2 & = & -25 & +x & -y & +t. \end{array}$$

Here, we have $-t$ in the objective row because minimizing t is equivalent to maximizing $-t$. Again, this dictionary is infeasible, but we may obtain a feasible dictionary with the new variable t. Variable t becomes basic, while a basic variable with the most negative value becomes non-basic. For the current dictionary, s_2 has the only variable with a negative value. Before moving t to the left-hand side, we apply row operations to eliminate t from the other rows.

$$\begin{array}{rcllll} z+s_2 &=& -25 & +x & -y & \\ s_1-s_2 &=& 175 & -3x & -2y & \\ s_2 &=& -25 & +x & -y & +t. \end{array}$$

Then we move t to the left-hand side and s_2 to the right-hand side.

$$\begin{array}{rcllll} z &=& -25 & +x & -y & -s_2 \\ s_1 &=& 175 & -3x & -2y & +s_2 \\ t &=& 25 & -x & +y & +s_2. \end{array}$$

Next, we note that x has a positive coefficient in the objective row. Then we may increase x up to

$$\min\left\{\frac{175}{3},\ 25\right\} = 25,$$

in which case t becomes 0 and becomes non-basic. Equivalently, we move x to the left-hand side and t to the right-hand side. Before this, we apply row operations as follows.

$$\begin{array}{rcllll} z+t &=& & & & \\ s_1-3t &=& 100 & & -5y & -2s_2 \\ t &=& 25 & -x & +y & +s_2. \end{array}$$

Then we deduce

$$\begin{array}{rcllll} z &=& & -t & & \\ s_1 &=& 100 & +3t & -5y & -2s_2 \\ x &=& 25 & -t & +y & +s_2. \end{array}$$

As all the coefficients in the objective row are nonpositive, this dictionary is optimal. Moreover, the optimal value is 0. Therefore, the original linear program is feasible, and the dictionary with basic variables s_1 and s provides a feasible dictionary. The feasible dictionary is given by

$$\begin{array}{rcll} s_1 &=& 100-5y & -2s_2 \\ x &=& 25+y & +s_2. \end{array}$$

Note that we are missing the objective row. In fact, the objective z is given by $z = x+2y$, and we can combine this with the dictionary. As x is a basic variable, we replace x with non-basic variables.

$$z = (25+y+s_2)+2y = 25+s_2+3y.$$

Then we obtain

$$\begin{array}{rclll} z &=& 25 & +s_2 & +3y, \\ s_1 &=& 100 & -2s_2 & -5y, \\ x &=& 25 & +s_2 & +y. \end{array}$$

The first feasible dictionary gives rise to the initial solution

$$(x,y) = (25,0) \quad (s_1,s_2) = (100,0).$$

2.2 Phase II: proceed the simplex method with the feasible dictionary

There are variables with positive coefficients in the objective row from the first feasible dictionary. In particular, y has a strictly positive coefficient. Let us make y basic. Then we may increase y up to 20, which would make s_1 non-basic. By applying the required row operations, we obtain

$$\begin{array}{rcllr} z + 0.6s_1 & = & 85 & -0.2s_2 & \\ s_1 & = & 100 & -2s_2 & -5y, \\ x + 0.2s_1 & = & 45 & +0.6s_2 & \end{array}$$

Moving s_1 to the right-hand side and y to the left-hand side, we obtain

$$\begin{array}{rclll} z & = & 85 & -0.2s_2 & -0.6s_1 \\ y & = & 20 & -0.4s_2 & -0.2s_1, \\ x & = & 45 & +0.6s_2 & -0.2s_1 \end{array}$$

Here, all objective coefficients are negative, and therefore, the currrent solution

$$(x, y) = (45, 20), \quad (s_1, s_2) = (0, 0)$$

is optimal.

2.3 Geometry

Note that the initial infeasible dictionary gave solution

$$(x, y) = (0, 0), \quad (s_1, s_2) = (150, -25).$$

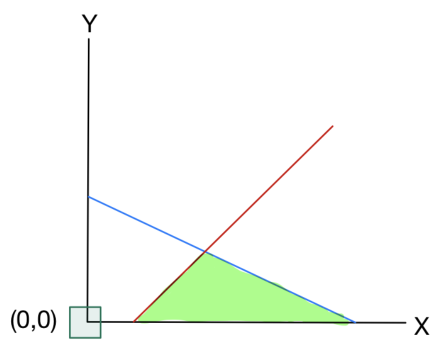

Figure 7.1: Solution from the initial infeasible dictionary

It can be checked from Figure 7.1 that $(x, y) = (0, 0)$ is an infeasible solution to the original linear program. Next, after Phase I, we obtained

$$(x, y) = (25, 0) \quad (s_1, s_2) = (100, 0)$$

from the first feasible dictionary. The first feasible solution is illustrated in the first figure in Figure 7.2. After this solution, we reached the optimality with solution

$$(x, y) = (45, 20), \quad (s_1, s_2) = (0, 0).$$

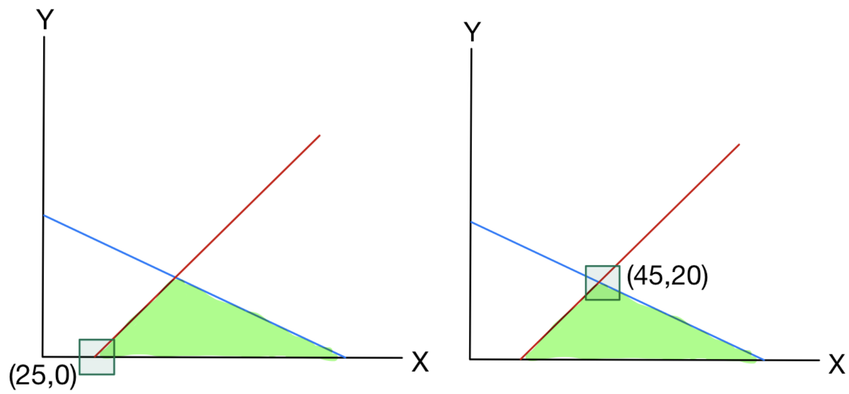

Figure 7.2: Solution from the first feasible dictionary and the optimal solution

3 Simplex method for infeasible and unbounded linear programs

3.1 Infeasible case

Next, we determine whether the following linear linear program is feasible or not.

$$\begin{aligned} \max \quad & z = x + 2y \\ \text{s.t.} \quad & 2x + 3y \leq 150, \\ & -x + y \leq -90, \\ & x, y \geq 0. \end{aligned}$$

As explained in the last section, we can test the feasibility by running Phase I. To do so, we consider

$$\begin{aligned} \min \quad & -t \\ \text{s.t.} \quad & 2x + 3y - t \leq 150, \\ & -x + y - t \leq -90, \\ & x, y \geq 0, \end{aligned}$$

which gives rise to the initial dictionary.

$$\begin{array}{rcllll} z & = & & & & -t, \\ s_1 & = & 150 & -2x & -3y & +t, \\ s_2 & = & -90 & +x & -y & +t. \end{array}$$

Moving t to the left-hand side and s_2 to the right-hand side, we obtain

$$\begin{array}{rcllll} z & = & -90 & +x & -y & -s_2 \\ s_1 & = & 240 & -3x & -2y & +s_2 \\ t & = & 90 & -x & +y & +s_2. \end{array}$$

Next x becomes basic while s_1 becomes non-basic. Applying row operations,

$$\begin{array}{rcllll} z + (1/3)s_1 & = & -10 & & -(5/3)y & -(2/3)s_2 \\ (1/3)s_1 & = & 80 & -x & -(2/3)y & +(1/3)s_2 \\ t - (1/3)s_1 & = & 10 & & +(5/3)y & +(2/3)s_2. \end{array}$$

Then we obtain

$$\begin{array}{rcllll} z &=& -10 & -(1/3)s_1 & -(5/3)y & -(2/3)s_2 \\ x &=& 80 & -(1/3)s_1 & -(2/3)y & +(1/3)s_2 \\ t &=& 10 & +(1/3)s_1 & +(5/3)y & +(2/3)s_2. \end{array}$$

Now all the objective coefficients are non-positive. However, the value of t is 10 which is strictly positive. Hence, the linear program is infeasible!

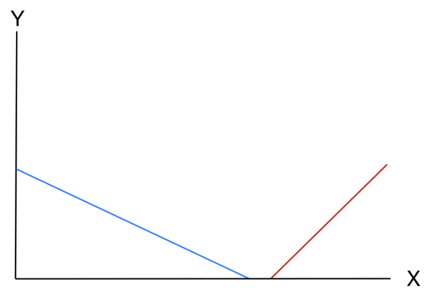

Figure 7.3: Infeasible case

3.2 Unbounded case

Now we consider

$$\begin{array}{rl} \max & z = x + y \\ \text{s.t.} & -2x + y \leq 100, \\ & x - 2y \leq 100, \\ & x, y \geq 0. \end{array}$$

Its standard form is given by

$$\begin{array}{rl} \max & z = x + y \\ \text{s.t.} & -2x + y + s_1 = 100, \\ & x - 2y + s_2 = 100, \\ & x, y \geq 0. \end{array}$$

Then the initial dictionary is given by

$$\begin{array}{rcllll} z &=& & +x & +y \\ s_1 &=& 100 & +2x & -y \\ s_2 &=& 100 & -x & +2y \end{array}$$

Next we make x basic and s_2 non-basic. Applying row operations,

$$\begin{array}{rclll} z + s_2 &=& 100 & & +3y \\ s_1 + 2s_2 &=& 300 & & +3y \\ s_2 &=& 100 & -x & +2y \end{array}$$

Moving s_2 to the right-hand side and x to the left-hand side, we deduce

$$\begin{array}{rclll} z &=& 100 & -s_2 & +3y \\ s_1 &=& 300 & -2s_2 & +3y \\ x &=& 100 & -s_2 & +2y \end{array}$$

Now, note that y has a positive objective coefficient. In fact, we may increase y by as much as we want without violating the nonnegativity constraints on the current basic variables! This implies that the linear program is unbounded.

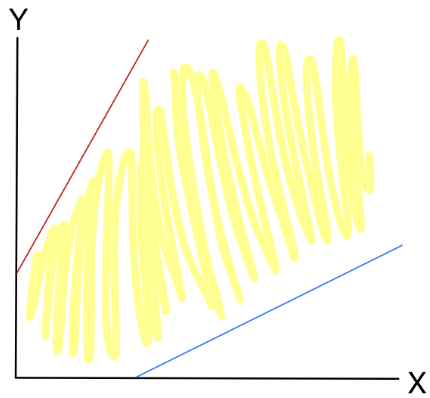

Figure 7.4: Unbounded case