1 Outline

In this lecture, we cover

- geometry of linear programming,
- a basic case of the simplex method,
- geometry of the simplex method.

2 Geometry of linear programming

Given a linear inequality $a^\top x \leq b$, the set of points satisfying the inequality, given by

$$\left\{x \in \mathbb{R}^d : \ a^\top x \leq b\right\}$$

is called a **half-space**. The set of points satisfying the inequality at equality, given by

$$\left\{x \in \mathbb{R}^d : \ a^\top x = b\right\},$$

is called a **hyperplane**.

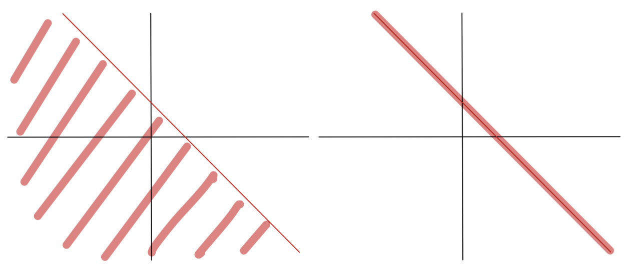

Figure 6.1: A half-space (left) and a hyperplane (right)

Definition 6.1. A set $C \subseteq \mathbb{R}^d$ is a **polyhedron** if it is defined by a **finite** number of linear inequalities, i.e.

$$C = \{x \in \mathbb{R}^d : \ a_i^\top x \leq b_i, \ \forall i \in [m]\}.$$

Furthermore, if C is bounded, i.e. there exists some $M > 0$ such that $C \subseteq \{x \in \mathbb{R}^d : \ -M \leq x_j \leq M, \ \forall j \in [d]\}$, then we say that C is a **polytope**.

By definition, a polyhedron is the intersection of a finite number of half-spaces. Figure 6.2 illustrates a polyhedron in $\mathbb{R}^2$ that is the intersection of three half-spaces.

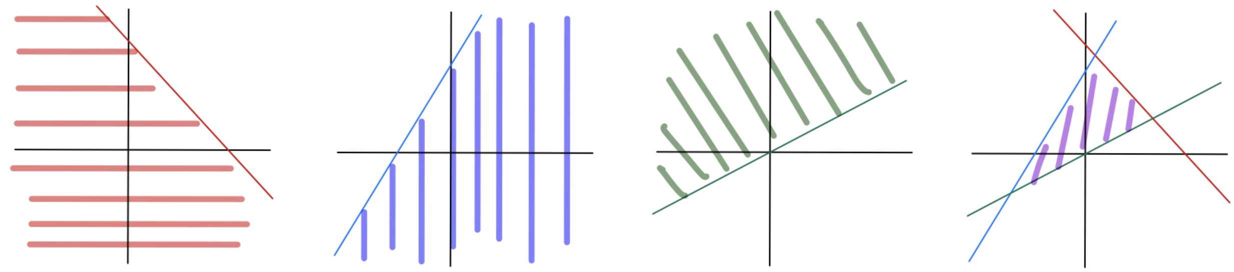

Figure 6.2: Polyhedron defined by three inequalities

Imagine a linear program that consists of only linear inequality constraints. As the number of constraints is finite, the set of feasible solutions is a polhedron. For example, a two-variable linear program

$$\begin{aligned} \max \quad & 5x + 4y \\ \text{s.t.} \quad & 2x + 3y \le 150, \\ & 2x + y \le 70, \\ & x, y \ge 0, \end{aligned}$$

has feasible region as in Figure 6.3. The polyhedron in Figure 6.3 is defined by 4 inequalities,

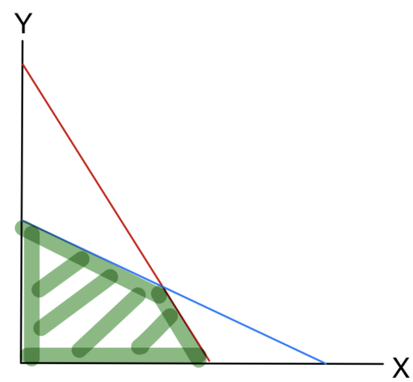

Figure 6.3: Feasible region of the two-variable linear program

$2x + 3y \le 150$, $2x + y \le 70$, $x \ge 0$, and $y \ge 0$.

Remark 6.2. Remember that the epigraph epi(f) of a linearly representable function f is given by

$$\text{epi}(f) = \left\{ (x,t) \in \mathbb{R}^d \times \mathbb{R} : \ \exists y \in \mathbb{R}^p \text{ s.t. } Ax + Dy + ht \le r \right\}$$

Moreover, the epigraph epi(f) is the projection of

$$P = \left\{ (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R} : \ Ax + Dy + ht \le r \right\}$$

onto the (x,t)-space. By definition, the set P is a polyhedron, and epi(f) is a projection of P. In fact, It is known that a projection of a polyhedron is also a polyhedron. This implies that there exists a representation of epi(f) in the original (x,t) space.

$$\text{epi}(f) = \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \ A'x + h't \leq r'\right\}$$

for some A', h', r'. Hence, if f is linearly representable, then its epigraph is a polyhedron.

Example 6.3. Remember that when $f(x) = \|x\|_1$,

$$\text{epi}(f) = \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \ \exists s \in \mathbb{R}^d \text{ s.t. } -s_j \leq x_j \leq s_j, \ \forall j \in [d], \ \sum_{j=1}^{d} s_j \leq t\right\}.$$

It is known that the description of epi(f) in the (x,t)-space is given by

$$\text{epi}(f) = \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \ \pi^\top x \leq t, \ \forall \pi \in \{-1,1\}^d\right\}.$$

Note that epi(f) is not bounded because we can set the value of t for a point $(x,t) \in \text{epi}(f)$ arbitrarily large. This means that epi(f) is a polyhedron while it is not a polytope.

3 Simplex algorithm

The simplex method due to George B. Dantzig is a practical algorithm for solving linear programming. The modern optimization software tools for linear and integer programming all implement and utilize the simplex method. We briefly touch upon the algorithm in this lecture.

Let us consider the two-variable linear program as follows. For running the simplex algorithm, it is common to introduce an extra variable z to represent the objective function.

$$\begin{aligned} \max \quad & z = 5x + 4y \\ \text{s.t.} \quad & 2x + 3y \leq 150, \\ & 2x + y \leq 70, \\ & x, y \geq 0. \end{aligned}$$

Step 1: represent the linear program in standard form. The first step is to write down the linear program in standard form. To write the two-variable LP in standard form, we use slack variables s_1 and s_2 for the constraints $2x+3y \leq 150$ and $2x + y \leq 70$, respectively. Then we obtain

$$\begin{aligned} \max_{x,y} \quad & z = 5x + 4y \\ \text{s.t.} \quad & 2x + 3y + s_1 = 150, \\ & 2x + y + s_2 = 70, \\ & x, y, s_1, s_2 \geq 0. \end{aligned}$$

Step 2: construct the initial dictionary. We keep the slack variables on the left-hand side and move the others to the right-hand side as follows.

$$\begin{array}{rcrrr} z & = & & +5x & +4y, \\ s_1 & = & 150 & -2x & -3y, \\ s_2 & = & 70 & -2x & -y. \end{array}$$

Here, the initial solution is given by setting the variables on the right-hand side to 0. In that case the values of the variables on the left-hand side are automatically chosen to satisfy the constraints. Therefore, the initial solution is given by

$$(x, y) = (0, 0), \quad (s_1, s_2) = (150, 70).$$

Then the corresponding objective value is

$$z = 5x + 4y = 0.$$

For an arbitrary linear program in standard form, we choose m distinct variables where m is the number of linear equality constraints. We keep these m variables on the left-hand side and the others on the right-hand side. We call the m variables **basic variables** and the others **non-basic variables**. To separate the basic variables from the non-basic variables, we may need a proper transformation.

Step 3: choose a better dictionary. Note that we have $z = 5x + 4y$ where non-basic variables are currently set to 0. Note that the coefficients of x and y are strictly positive. This means that increasing x or y would increase the objective value z. It looks like that the rate of increase is higher with x than y, because x has a larger objective coefficient. This is the step of

- deciding the non-basic variable to become a new basic variable (choosing the entering variable).

Let us try to increase x while the other non-basic variable, y, remains 0. Consider

$$\begin{array}{rcll} s_1 &=& 150 & -2x, \\ s_2 &=& 70 & -2x. \end{array}$$

We still want the current basic variables to be nonnegative. While keeping both s_1 and s_2 nonnegative, we may increase x up to

$$\min\left\{\frac{150}{2}, \frac{70}{2}\right\}.$$

As the value is 35, we may increase x up to 35, in which case s_1 becomes 80 while s_2 becomes 0. As the value of s_2 is now 0, we may switch the roles of x and s_2 and move s_2 to the right-hand side. This is the step of

- deciding the basic variable to become a non-basic variable (choosing the leaving variable).

Before we move x to the left-hand side, we do some matrix row operations to eliminate x from the rows not containing variable s_2. Then we obtain

$$\begin{array}{rclll} z + 2.5s_2 &=& 175 & & +1.5y, \\ s_1 - s_2 &=& 80 & & -2y, \\ s_2 &=& 70 & -2x & -y. \end{array}$$

Next we move s_2 to the right-hand side and x to the left-hand side.

$$\begin{array}{rclll} z &=& 175 & -2.5s_2 & +1.5y, \\ s_1 &=& 80 & +s_2 & -2y, \\ x &=& 35 & -0.5s_2 & -0.5y. \end{array}$$

Then s_1 and x are now basic variables while s_2 and y are non-basic variables. Then we obtain a new solution given by

$$(x, y) = (35, 0), \quad (s_1, s_2) = (80, 0).$$

In this case, the objective value is

$$z = 175 - 2.5s_2 + 1.5y = 175.$$

Step 4: repeat step 3. We have $z = 175 - 2.5s_2 + 1.5y$ and y is set to 0. Then we may still have some room for improvement by increasing the value of y. Then while keeping s_2 zero, we attempt to increase y. To decide how much we increase y, consider

$$\begin{array}{rcll} s_1 & = & 80 & -2y, \\ x & = & 35 & -0.5y. \end{array}$$

While keeping s_1 and x nonnegative, we may increase y up to 40. In this case, s_1 becomes 0 and x becomes 15. Then applying the desired row operations,

$$\begin{array}{rclll} z + 0.75s_1 & = & 235 & -1.75s_2 & \\ s_1 & = & 80 & +s_2 & -2y, \\ x - 0.25s_1 & = & 15 & -0.75s_2 & \end{array}$$

Then moving y to the left-hand side and s_1 to the right-hand side, we obtain

$$\begin{array}{rcllll} z & = & 235 & -1.75s_2 & -0.75s_1 \\ y & = & 40 & +0.5s_2 & -0.5s_1, \\ x & = & 15 & -0.75s_2 & +0.25s_1 \end{array}$$

Then we obtain a new solution given by

$$(x, y) = (15, 40), \quad (s_1, s_2) = (0, 0).$$

In this case, the objective value is

$$z = 235 - 1.75s_2 - 0.75s_1 = 235.$$

Step 5: obtain an optimal solution. Now, we represent the objective as

$$z = 35 - 1.75s_2 - 0.75s_1$$

where the current non-basic variables are s_1 and s_2. The coefficients of s_1 and s_2 are both negative. Then increasing the value of s_1 or that of s_2 would deteriorate the objective. In fact, this means that we are currently at an optimal solution!

4 Geometry of the simplex algorithm

Remember that we have taken 3 solutions until we solve the two-variable linear program.

1. $(x, y) = (0, 0)$ and $(s_1, s_2) = (150, 70)$,
2. $(x, y) = (35, 0)$ and $(s_1, s_2) = (80, 0)$,
3. $(x, y) = (15, 40)$ and $(s_1, s_2) = (0, 0)$.

Let us consider the first solution given by $(x, y) = (0, 0)$ and $(s_1, s_2) = (150, 70)$. What is the point of setting $x = y = 0$ and assigning positive values fo the slack variables s_1 and s_2? Note that s_1 being positive means $2x + 3y < 150$ and the current point (x, y) is not on the line defined by $2x + 3y = 150$. Similarly, positive s_2 means that the point (x, y) is not on the line defined by $2x + y < 70$. In fact, $(x, y) = (0, 0)$ satisfies the constraints $x \geq 0$ and $y \geq 0$ at equality, and the point is at the intersection of line $x = 0$ and line $y = 0$.

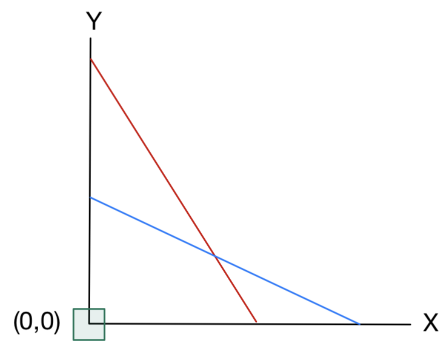

Figure 6.4: Illustrating the initial solution

The second solution has $(x, y) = (35, 0)$ and $(s_1, s_2) = (80, 0)$. Note that the point is on the line $y = 0$. Moreover, s_1 is still positive, and we have $2x + 3y = 70 < 150$. Hence, the first slack variable being strictly positive means that the constraint $2x + 3y \leq 150$ is not tight. In contrast, the other slack variable is $s_2 = 0$. Note that $2x + y = 70$, so the second constraint is satisfied at equality.

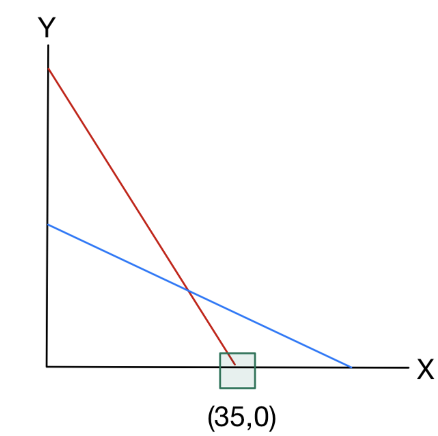

Figure 6.5: Illustrating the second solution

The third solution has $(x, y) = (15, 40)$ and $(s_1, s_2) = (0, 0)$. Note that both x and y are strictly positive, while the slack variables are all 0. We can check that $(x, y) = (15, 40)$ satisfies the both constraints $2x + 3y \leq 150$ and $2x + y \leq 70$ at equality. In fact the point is at the intersection of the two hyperplanes $2x + 3y = 150$ and $2x + y = 70$.

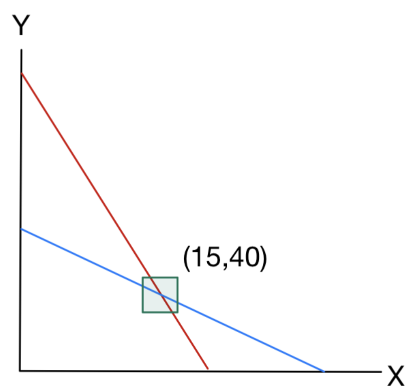

Figure 6.6: Illustrating the third solution