1 Outline

In this lecture, we cover

- the linear programming formulation for the inventory valuation problem,
- linearly representable functions,
- representing optimization problems as linear programs.

2 Optimization problems represented as linear programs

2.2 Projection

Remember that f is linearly representable if its epigraph $\mathrm{epi}(f)$ is given by

$$\mathrm{epi}(f) = \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \ \exists y \in \mathbb{R}^p \text{ s.t. } Ax + Dy + ht \leq r\right\}$$

for some $A \in \mathbb{R}^{\ell \times d}$, $D \in \mathbb{R}^{\ell \times p}$, and $h, r \in \mathbb{R}^\ell$. Here, what does it mean by having auxiliary variables y?

To answer this, we discuss the concept of **projection**. Let $C \subseteq \mathbb{R}^d \times \mathbb{R}^p$ be a set in the space $\mathbb{R}^d \times \mathbb{R}^p$. For convenience, we represent each point in C as (x, y) where $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^p$. Then the projection of C onto the space of x part[1], which is $\mathbb{R}^d$, is given by

$$\mathrm{proj}_x(C) = \left\{x \in \mathbb{R}^d : \ \exists y \in \mathbb{R}^p \text{ s.t. } (x,y) \in C\right\}.$$

We also refer to this operation as **projecting out the y variables**. Figure 4.1 illustrates the projection of a set in $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

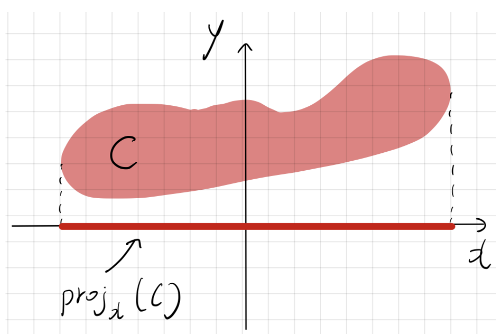

Figure 4.1: Projection of a set in $\mathbb{R}^2$ to $\mathbb{R}$

Getting back to $\mathrm{epi}(f)$, we take the set of vectors that have the (x, y, t) part and obtain

$$P = \left\{(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R} : \ Ax + Dy + ht \leq r\right\}.$$

[1]This may not be the most conventional terminology.

Then epi(f) is the projection of P onto the (x,t)-space.

2.3 Common linearly representable functions

In the previous section, we saw some linearly representable functions. In this section, we conver more linearly representable functions.

Absolute value Let $f(x) = |x|$ for $x \in \mathbb{R}$. Note that $|x| \leq t$ if and only if $-t \leq x \leq t$. Therefore,

$$\text{epi}(f) = \{(x,t) \in \mathbb{R} \times \mathbb{R} : \ -t \leq x \leq t\}.$$

Convex piecewise linear functions Let $f(x) = \max_{i\in[n]} \left\{c_i^\top x + d_i\right\}$. Note that $f(x) = \max_{i\in[n]} \left\{c_i^\top x + d_i\right\} \leq t$ if and only if $c_i^\top x + d_i \leq t$ for $i \in [n]$. Then

$$\text{epi}(f) = \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \ c_i^\top x + d_i \leq t, \quad \forall i \in [n]\right\}.$$

ℓ_1-norm Let $f(x) = \|x\|_1 = \sum_{j\in[d]} |x_j|$. Note that $\|x\|_1 \leq t$ if and only if there exist $s_1, \ldots, s_d \in \mathbb{R}$ such that

$$|x_j| \leq s_j, \ \forall j \in [d], \quad \sum_{j\in[d]} s_j \leq t.$$

This is equivalent to

$$-s_j \leq x_j \leq s_j, \ \forall j \in [d], \quad \sum_{j\in[d]} s_j \leq t.$$

ℓ_∞-norm Let $f(x) = \|x\|_\infty = \max_{j\in[d]} |x_j|$. Note that $\|x\|_\infty \leq t$ if and only if

$$|x_j| \leq t \ \forall j \in [d]$$

This is equivalent to

$$-t \leq x_j \leq t, \ \forall j \in [d].$$

In fact, there are some operations that preserve linear representability, with which we can build more complex linearly representable functions.

Theorem 4.1. *Let $f_1(x), \ldots, f_n(x)$ be linearly representable functions. Then the following statements hold.*

1. *For any non-negative scalars $\alpha_1, \ldots, \alpha_n$,*

$$f(x) = \sum_{i\in[n]} \alpha_i f_i(x)$$

   is linearly representable,

2. *The point-wise maximum of $f_1, \ldots, f_n$, given by*

$$f(x) = \max_{i\in[n]} f_i(x)$$

   is linearly representable.

Proof. For the first part,

$$\begin{aligned}
\mathrm{epi}(f) &= \left\{(x,t)\in\mathbb{R}^d\times\mathbb{R}:\ \sum_{i\in[n]}\alpha_i f_i(x)\le t\right\}\\
&= \left\{(x,t)\in\mathbb{R}^d\times\mathbb{R}:\ \exists s\in\mathbb{R}^n \text{ s.t. } \sum_{i\in[n]}\alpha_i s_i\le t,\ f_i(x)\le s_i,\ \forall i\in[n]\right\}\\
&= \left\{(x,t)\in\mathbb{R}^d\times\mathbb{R}:\ \exists s\in\mathbb{R}^n \text{ s.t. } \sum_{i\in[n]}\alpha_i s_i\le t,\ (x,s_i)\in\mathrm{epi}(f_i),\ \forall i\in[n]\right\}.
\end{aligned}$$

For the second part,

$$\begin{aligned}
\mathrm{epi}(f) &= \left\{(x,t)\in\mathbb{R}^d\times\mathbb{R}:\ \max_{i\in[n]} f_i(x)\le t\right\}\\
&= \left\{(x,t)\in\mathbb{R}^d\times\mathbb{R}:\ f_i(x)\le t,\ \forall i\in[n]\right\}\\
&= \left\{(x,t)\in\mathbb{R}^d\times\mathbb{R}:\ (x,t)\in\mathrm{epi}(f_i),\ \forall i\in[n]\right\}.
\end{aligned}$$

In both cases, $\mathrm{epi}(f)$ is represented by a finite system of linear inequalities because $f_1,\ldots,f_n$ are linearly representable. $\square$

3 Production planning with holding costs

In this section, we consider yet another variant of the production planning problem. Namely, we consider **holding costs** of products. Consider the following setup for the production planning problem.

- There are d different products and m different kinds of raw materials necessary for producing the products.
- One unit of product j sells for price p_j per unit for $i\in[d]$.
- The current stock of material i is b_i for $i\in[m]$.
- Producing one unit of product j requires a_{ij} amount of material i for every pair $(i,j)\in[m]\times[d]$. Here, $[m]\times[d]$ denote the set $\{(i,j): i\in[m], j\in[d]\}$.
- Each product $j\in[d]$ has a prescribed demand d_j, and the company can sell up to only d_j units.
- If the company produces more than d_j units, then extra production incurs c_j holding cost per unit.
- If the company produces less than d_j units, then unsatisfied demand incurs h_j penalty cost per unit.
- Assume that there is no other cost.

As before, we use variable x_j for the amount of product $j \in [d]$. Then we have $x_j \geq 0$ for $j \in [d]$, and

$$\sum_{j \in [d]} a_{ij} x_j \leq b_i$$

due to the current stock of raw material $i \in [m]$. To take holding costs into the objective, we observe the following. If $x_j \geq d_j$, then the compnay sells precisely d_j units and the rest corresponds to excess production. The holding cost due to excess production is $c_j(x_j - d_j)$ while there is no penalty. Hence, the total profit from producing x_j units of product j is

$$\underbrace{p_j d_j}_{\text{selling at most } d_j \text{ units}} - \underbrace{c_j(x_j - d_j)}_{\text{holding cost}} = -c_j x_j + (p_j + c_j) d_j.$$

If $x_j < d_j$, then the demand is not fully satisfied, and the company sells all the x_j units produced. There is no holding cost, while the penalty from unsatisfied demand is $h_j(d_j - x_j)$. Hence, the total profit from producing x_j units of product j is

$$\underbrace{p_j x_j}_{\text{selling } x_j \text{ units}} - \underbrace{h_j(d_j - x_j)}_{\text{penalty cost}} = (p_j + h_j) x_j - h_j d_j.$$

Figure 4.2 shows the profit function for product j. As the production increases from j up to d_j, the profit increases with the rate of $(p_j + h_j)$. When the production goes beyond d_j, the profit starts decreasing with the rate of c_j.

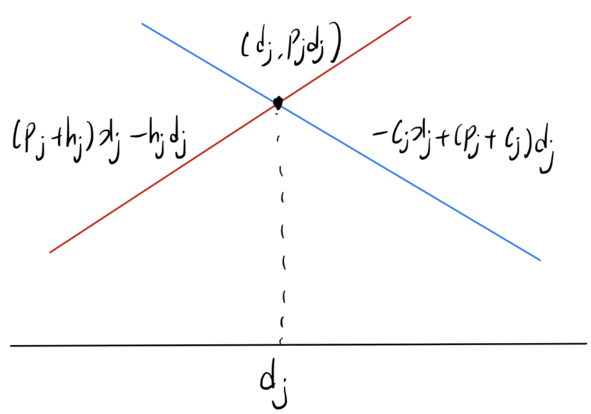

Figure 4.2: Profit function

Based on this, the profit function of product j is given by

$$\min \{-c_j x_j + (p_j + c_j) d_j, \ (p_j + h_j) x_j - h_j d_j\}.$$

Hence, the objective we wish to maximize is the profit sum of products, given by

$$f(x) = \sum_{j \in [d]} \min \{-c_j x_j + (p_j + c_j) d_j, \ (p_j + h_j) x_j - h_j d_j\}.$$

Therefore, the optimization model for the production planning problem is given by

$$\begin{aligned}
\max \quad & \sum_{j\in[d]} \min\left\{-c_j x_j + (p_j + c_j)d_j,\ (p_j + h_j)x_j - h_j d_j\right\} \\
\text{s.t.} \quad & \sum_{j\in[d]} a_{ij}x_j \le b_i, \quad \forall i \in [m], \\
& x_j \ge 0, \quad \forall j \in [d].
\end{aligned} \tag{4.1}$$

Next we convert this into a linear program. First, we may use the theorem on linear representability. As the constraints are already given by linear functions, we need to show that the objective is linearly representable. As the theorem is for the minimization problem, we can convert the maximization problem into the equivalent minimization problem, given as follows.

$$\begin{aligned}
\min \quad & -\sum_{j\in[d]} \min\left\{-c_j x_j + (p_j + c_j)d_j,\ (p_j + h_j)x_j - h_j d_j\right\} \\
\text{s.t.} \quad & \sum_{j\in[d]} a_{ij}x_j \le b_i, \quad \forall i \in [m], \\
& x_j \ge 0, \quad \forall j \in [d],
\end{aligned}$$

and this is equivalent to

$$\begin{aligned}
\min \quad & \sum_{j\in[d]} \max\left\{c_j x_j - (p_j + c_j)d_j,\ -(p_j + h_j)x_j + h_j d_j\right\} \\
\text{s.t.} \quad & \sum_{j\in[d]} a_{ij}x_j \le b_i, \quad \forall i \in [m], \\
& x_j \ge 0, \quad \forall j \in [d],
\end{aligned}$$

Here, the objective is the sum of convex piecewise linear functions, so it is linearly representable. The technique we learned from the last lecture gives us

$$\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & \sum_{j\in[d]} s_j \le t, \\
& c_j x_j - (p_j + c_j)d_j \le s_j, \quad \forall j \in [d], \\
& -(p_j + h_j)x_j + h_j d_j \le s_j, \quad \forall j \in [d], \\
& \sum_{j\in[d]} a_{ij}x_j \le b_i, \quad \forall i \in [m], \\
& x_j \ge 0, \quad \forall j \in [d].
\end{aligned}$$

The second approach is to directly convert the maximization problem. Note that maximizing $f(x)$ is equivalent to maximizing an auxiliary variable t subject to $f(x) \ge t$. Here $f(x) \ge t$ is

$$\sum_{j\in[d]} \min\left\{-c_j x_j + (p_j + c_j)d_j,\ (p_j + h_j)x_j - h_j d_j\right\} \ge t,$$

which holds if and only if there exist $s_1, \ldots, s_d \in \mathbb{R}$ such that

$$\sum_{j\in[d]} s_j \ge t, \quad \min\left\{-c_j x_j + (p_j + c_j)d_j,\ (p_j + h_j)x_j - h_j d_j\right\} \ge s_j,\ \forall j \in [d].$$

Here, $\min\{-c_jx_j + (p_j + c_j)d_j,\ (p_j + h_j)x_j - h_jd_j\} \geq s_j$ if and only if

$$-c_jx_j + (p_j + c_j)d_j \geq s_j, \quad (p_j + h_j)x_j - h_jd_j \geq s_j.$$

Hence, we deduce the linear program

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & t \leq \sum_{j\in[d]} s_j, \\ & s_j \leq -c_jx_j + (p_j + c_j)d_j, \quad \forall j \in [d], \\ & s_j \leq (p_j + h_j)x_j - h_jd_j, \quad \forall j \in [d], \\ & \sum_{j\in[d]} a_{ij}x_j \leq b_i, \quad \forall i \in [m], \\ & x_j \geq 0, \quad \forall j \in [d]. \end{aligned} \tag{4.2}$$

In fact, the formulations (4.1) and (4.2) are equivalent, as we may relabel $t \to -t$ and $s_j \to -s_j$ for $j \in [d]$.