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1 Outline

In this lecture, we cover

- the linear programming formulation for the inventory valuation problem,
- linearly representable functions,
- representing optimization problems as linear programs.

2 Inventory valuation problem

Consider the same company from the production planning problem.

- There are d different products and m different kinds of raw materials necessary for producing the products.
- One unit of product j sells for price p_j for $i \in [d]$.
- The current stock of material i is b_i for $i \in [m]$. **We assume that each b_i is strictly positive! If $b_i = 0$ for some $i \in [m]$, we would not be able to make product j with $a_{ij} > 0$.**
- Producing one unit of project j requires a_{ij} amount of material i for every pair $(i, j) \in [m]\times[d]$. Here, $[m] \times [d]$ denote the set $\{(i, j) : i \in [m], j \in [d]\}$.

Imagine a situation where the company could sell out the material inventory instead of consuming it to make products. Assume that

- the current market price of material i is r_i per unit for $i \in [m]$.

The company tries to evaluate how much the current raw material inventory worth to the company. Here, the value of raw material i is measured based on its market price and the prices of end products obtained from consuming it.

We can solve this valuation problem by the following linear program. Let λ_i denote the unit value of material i that we want to decide. First, we should satisfy

$$\lambda_i \geq r_i$$

for each material $i \in [m]$ because we have the option of selling out the material. Second, for each product $j \in [d]$, we should satisfy

$$\sum_{i\in[m]} a_{ij}\lambda_i \geq p_j$$

because we can make product j that sells for price p_j per unit. Being as conservative as possible, the total value of b_i units of material i for $i \in [m]$ can be measured by the following linear program.

$$
\begin{aligned}
\min \quad & \sum_{i\in[d]} b_i\lambda_i \\
\text{s.t.} \quad & \sum_{i\in[m]} a_{ij}\lambda_i \geq p_j, \quad j \in [d], \\
& \lambda_i \geq r_i, \quad i \in [m].
\end{aligned}
$$

3 Optimization problems represented as linear programs

In general, what type of decision-making problems can be modeled as linear programs? Recall that an optimization problem is of the form

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leq b_i, \quad i \in [m], \\
& x \in \mathbb{R}^d.
\end{aligned}
$$

Here, if the objective function f and the constraint functions $g_1, \ldots, g_m$ are linear, then the optimization problem is a linear program. In fact, even when the functions are non-linear, we may still be able to represent the problem as a linear program.

Consider the following problem. Let $V = \{v^1, \ldots, v^n\} \subseteq \mathbb{R}^d$ be a set of vectors in a cluster. The problem is to determine the center of the cluster. Here, the center is a point, from which the distance to the set of n vectors is minimized. Depending on which distance metric is used, we may obtain different centers.

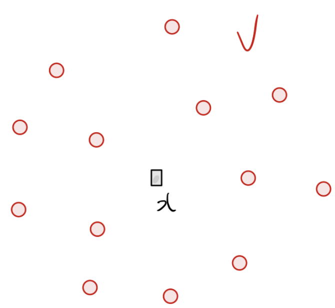

Figure 3.1: Cluster of points

To define certain distance metrics, we consider some vector norms. Given a vector $x \in \mathbb{R}^d$, the ℓ_1**-norm** is defined as

$$\|x\|_1 = \sum_{j\in[d]} |x_j|,$$

and the ℓ_∞**-norm** is defined as

$$\|x\|_\infty = \max_{j\in[d]} |x_j|.$$

Based on the vector norms, we will define the distance between a point x and the set V, denoted as $d(x)$, and we set the center of the cluster by solving

$$\min_x \quad d(x).$$

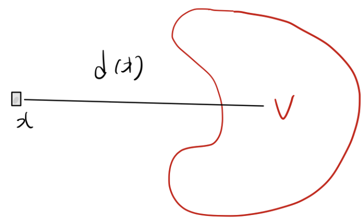

Figure 3.2: $d(x)$: the distance between x and the set V

Now we consider the following four distance functions.

1. The sum of the ℓ_1-distance between x and individual vector v^i for $i \in [n]$.

$$d(x) = \sum_{i \in [n]} \left\| x - v^i \right\|_1 .$$

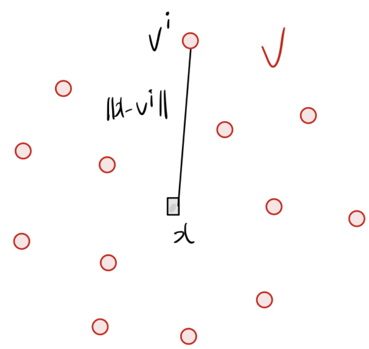

Figure 3.3: The norm-based distance between x and individual v^i

2. The sum of the ℓ_∞-distance between x and individual vector v^i for $i \in [n]$.

$$d(x) = \sum_{i \in [n]} \left\| x - v^i \right\|_\infty .$$

3. The maximum ℓ_1-distance between x and individual vector v^i for $i \in [n]$.

$$d(x) = \max_{i \in [n]} \left\| x - v^i \right\|_1 .$$

4. The maximum ℓ_∞-distance between x and individual vector v^i for $i \in [n]$.

$$d(x) = \max_{i \in [n]} \left\| x - v^i \right\|_\infty .$$

Here, as the vector norms are not linear, the distance functions are not linear either. Nevertheless, we will see that the problem for each of the four distance functions can be rewritten as a linear program.

3.1 Linearly representable functions

In this section, we study when an optimization problem can be recast as a linear program. The **epigraph** of a function $f : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\mathrm{epi}(f) = \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \ f(x) \le t\} \subseteq \mathbb{R}^{d+1}.$$

For example, Figure 3.4 illustrates the epigraph of a function over $\mathbb{R}$.

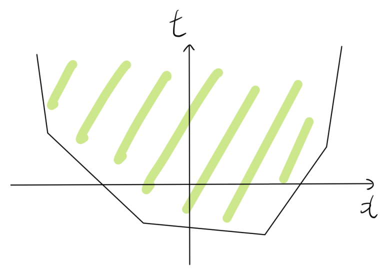

Figure 3.4: Epigraph of a function

A function $f : \mathbb{R}^d \to \mathbb{R}$ is **linearly representable** if its epigraph $\mathrm{epi}(f)$ is represented with a **finite** system of linear inequalities, i.e.,

$$\mathrm{epi}(f) = \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \ \exists y \in \mathbb{R}^p \text{ s.t. } Ax + Dy + ht \le r\right\}$$

for some $A \in \mathbb{R}^{\ell \times d}$, $D \in \mathbb{R}^{\ell \times p}$, and $h, r \in \mathbb{R}^\ell$. Here, the variables y in the linear representation are called **auxiliary variables**. We want variables (x, y, t) to satisfy the linear inequalities

$$Ax + Dy + ht \le r,$$

but $\mathrm{epi}(f)$ collects the points that has the (x,t) part.

Theorem 3.1. *The optimization problem*

$$\min \{f(x) : \ g_i(x) \le b_i, \ \forall i \in [m]\}$$

can be reformulated as a linear program if the objective function and the constraint functions are linearly representable.

Proof. The objective is to minimize $f(x)$. We may reformulate this by introducing an auxiliary variable $t \in \mathbb{R}$. To be specific, minimizing $f(x)$ is equivalent to minimizing t subject to an additional constraint $f(x) \le t$. Therefore, the optimization is equivalent to

$$\min \{t : \ f(x) \le t, \ g_i(x) \le b_i, \ \forall i \in [m]\}.$$

Using the notion of epigraphs, we can equivalently write this as

$$\min \{t : \ (x,t) \in \mathrm{epi}(f), \ (x, b_i) \in \mathrm{epi}(g_i), \ \forall i \in [m]\}.$$

Since f is linearly representable,

$$\mathrm{epi}(f) = \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \ \exists y \in \mathbb{R}^p \text{ s.t. } Ax + Dy + ht \le r\right\}$$

for some $A \in \mathbb{R}^{\ell \times d}$, $D \in \mathbb{R}^{\ell \times p}$, and $h, r \in \mathbb{R}^{\ell}$. Moreover, since $g_1, \ldots, g_m$ are linearly representable,

$$\mathrm{epi}(g_i) = \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} :\ \exists z^i \in \mathbb{R}^{q_i} \text{ s.t. } A^i x + D^i z^i + h^i t \leq r^i\right\}$$

for some $A^i \in \mathbb{R}^{\ell_i \times d}$, $D^i \in \mathbb{R}^{\ell_i \times p}$, and $h^i, r^i \in \mathbb{R}^{\ell_i}$. Therefore, the optimization problem is equivalent to

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & Ax + Dy + ht \leq r, \\ & A^i x + D^i z^i \leq r^i - h^i b_i, \quad i \in [m], \\ & x \in \mathbb{R}^d,\ t \in \mathbb{R},\ y \in \mathbb{R}^p,\ z_i \in \mathbb{R}^{q_i},\ i \in [m], \end{aligned}$$

which is a linear program. □

By definition, any linear function is linearly representable. In fact, there exist functions that are non-linear but linearly representable.

Example 3.2. $f(x_1, x_2, x_3) = \max\{x_1,\ 2x_2 + x_3,\ 2x_3 - x_1\}$. Its epigraph is given by

$$\{(x_1, x_2, x_3, t) :\ \max\{x_1,\ 2x_2 + x_3,\ 2x_3 - x_1\} \leq t\}$$

Note that $\max\{x_1,\ 2x_2 + x_3,\ 2x_3 - x_1\} \leq t$ is equivalent to

$$x_1, \leq t,\ 2x_2 + x_3 \leq t,\ 2x_3 - x_1 \leq t.$$

Therefore, it follows that

$$\mathrm{epi}(f) = \{(x_1, x_2, x_3, t) :\ x_1 \leq t,\ 2x_2 + x_3 \leq t,\ 2x_3 - x_1 \leq t\},$$

so f is linearly representable.