Lecturer: Dabeen Lee

1 Outline

In this lecture, we cover

- a review of vectors and matrices,
- components optimization models,
- terminologies in mathematical optimization,
- linear functions and linear programming models.

2 Quick review of linear algebra

Throughout the course, we will frequently refer to vectors, matrices, and other parts of linear algebra. Just for today, let us briefly review the notions of vectors and matrices.

We write a **vector** $x \in \mathbb{R}^d$ in the d-dimensional space as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \quad \text{or} \quad x = (x_1, \ldots, x_d).$$

The first represents x as a **column vector**, while the second is the **row vector** representation. Rigorously speaking, they are the transpose of one another, i.e.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} = (x_1, \ldots, x_d)^\top,$$

but we interchageably use them depending on the context. $x_1, \ldots, x_d$ are called the **components** or **coordinates** of vector x.

An $m \times d$ matrix $A \in \mathbb{R}^{m\times d}$ is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{md} \end{bmatrix} = \begin{bmatrix} a_1^\top \\ a_2^\top \\ \vdots \\ a_m^\top \end{bmatrix}.$$

Here, a_{ij} for $i = 1, \ldots, m$ and $j = 1, \ldots, d$ are the **entries** of A. Moreover, $a_i \in \mathbb{R}^d$ for $i = 1, \ldots, m$ are called the **rows** of A and given by

$$a_i^\top = \begin{bmatrix} a_{i1} & a_{12} & \cdots & a_{id} \end{bmatrix} \quad \text{and} \quad a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{id} \end{bmatrix}.$$

Again, $a_i^\top$ is a row vector while a_i is a column vector. Those who are not familiar with vectors and matrices should review the basics of linear algebra.

3 Components of optimization models

A decision-making problem has the following three components.

1. **Decisions**: parameters and variables that we need to determine. For example, the number of units to produce, a schedule of sports games, and a regression model.
2. **Constraints**: restrictions and requirements that we need to satisfy. For example, the limits on how much material we use for production and balancing the number of away games and that of home games for a team.
3. **Objective**: what we are trying to achieve. Typically either maximizing profits or minimizing costs.

Optimization models formulate these three components of a decision-making problem mathematically and quantitatively.

- Decisions are encoded by variables $x = (x_1, \ldots, x_d)$ where each x_j has to assign a **numerical value**.
- Constraints are modeled by some functions of variables, i.e., a constraint is of the form $g(x) \leq b$ for some function $g : \mathbb{R}^d \to \mathbb{R}$ and some bound $b \in \mathbb{R}$. Here, without loss of generality, we may assume that $b = 0$ for a constraint because $g(x) \leq b$ is equivalent to $g(x) - b \leq 0$.
- The objective is a function $f : \mathbb{R}^d \to \mathbb{R}$ that want to maximize or minimize.

We may have multiple decisions and constraints but usually there is a single objective. Settings where the decision-maker has to consider multiple objectives can be formulated as the so-called *multi-objective optimization* problem, but throughout the course, we only consider problems with a single objective.

Given the formalism, we may state an optimization problem as follows.

> Choose variables x to make $f(x)$ as large (or small) as possible under the restrictions that x satisfy $g_1(x) \leq b_1, \ldots, g_m(x) \leq b_m$.

Standard ways to represent this is

$$\begin{aligned} \min_x / \max_x \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq b_i, \quad i \in [m] \end{aligned} \tag{2.1}$$

and

$$\min_x / \max_x \{f(x) : \ g_i(x) \leq b_i, \quad i \in [m]\} \tag{2.2}$$

Here, "s.t." stands for "subject to", and $[m]$ denotes the set $\{1, \ldots, m\}$. Note that there is a subscript x under min and max. It means that the optimization problem is over decision variables

x. Unless there is some constraint on x, e.g. $x \in \mathbb{Z}^d$ and $x \in [0,1]^d$, we assume that $x \in \mathbb{R}^d$, meaning that the components of x take real values. That said, (2.1) can be equivalently written as

$$\begin{aligned} \min / \max \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq b_i, \quad i \in [m], \\ & x \in \mathbb{R}^d. \end{aligned} \tag{2.3}$$

It is common to omit the constraint $x \in \mathbb{R}^d$.

Example 2.1. Let us get back to the production planning question. Let x and y be the variables for the number of product X and and that of Y, respectively. Recall that we have 150 units of

	A	B
X	2	2
Y	3	1

Table 1: Materials required to produce products

material A and 70 units of material B, which imposes the constraints $2x+3y \leq 150$ and $2x+y \leq 70$. Moreover, the quantities x and y cannot be negative. Lastly, the objective is to maximize the total number of products produced. Here, the number of produced products would be $x+y$. Hence, the problem can be formulated as

$$\begin{aligned} \max_{x,y} \quad & x + y \\ \text{s.t.} \quad & 2x + 3y \leq 150, \\ & 2x + y \leq 70, \\ & x, y \geq 0. \end{aligned} \tag{2.4}$$

4 Terminologies in mathematical optimization

Next we establish some conventional terminologies in the field of mathematical optimization. Let us take an optimization problem whose objective is to minimize some function.

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq b_i, \quad i \in [m], \\ & x \in \mathbb{R}^d. \end{aligned} \tag{2.5}$$

- $f(x)$ is the **objective function**.
- $g_1(x), \ldots, g_m(x)$ are **constraint functions**, and $b_1, \ldots, b_m$ are called **right-hand sides**.
- Any vector $x \in \mathbb{R}^d$ is a **solution**. A solution x satisfying all the constraints, $g_i(x) \leq b_i$ for $i \in [m]$ is called **feasible** or a **feasible solution**. If a solution x violates some constraint, i.e. $g_i(x) > b_i$ for some $i \in [m]$, we call x **infeasible** or an **infeasible solution**.
- The value $p^* = \min_x \{f(x) : \ g_i(x) \leq b_i, \quad i \in [m]\}$, as long as it is finite, is the **optimal value**. We also call it the **optimum**.
- Given a solution x, we call $f(x)$ the **objective value** of x.

- A feasible solution x^* whose objective value $f(x^*)$ is equal to the optimal value p^* is called **optimal** or an **optimal solution**. In fact, there can be more than one optimal solutions.
- We say that the optimization problem (2.5) is **feasible** if it admits a feasible solution. We say that the problem is **infeasible** if there is no feaasible solution, in which case, we set $p^* = \infty$ by convention.
- If, for any $r \in \mathbb{R}$, there is a feasible solution x_r whose objective $f(x_r)$ is less than r, then the problem is **unbounded**, in which case, we set $p^* = -\infty$.

The definitions and terminologies apply to maximization problems, as a maximization problem can be equivalently transformed into a minimization problem. Note that

$$\begin{aligned}&\max_x \{f(x) : \; g_i(x) \le b_i, \quad i \in [m]\}\\&= -\min_x \{-f(x) : \; g_i(x) \le b_i, \quad i \in [m]\}.\end{aligned}$$

Here, the maximization problem being infeasible means that $p^* = -\infty$ while being unbounded means $p^* = \infty$.

Example 2.2. In problem (2.4), any vector $(x, y) \in \mathbb{R}^2$ is a solution. Note that $(0, 0)$, $(10, 10)$, $(20, 20)$ all satisfy the constraints, so they are feasible solutions. However, $(30, 30)$ violates the second constraint as $2 \times 30 + 30 = 90 > 70$, so $(30, 30)$ is an infeasible solution.

As problem (2.4) has a feasible solution, the problem is feasible. We have seen that $(15, 40)$ is an optimal solution and that $15 + 40 = 55$ is the optimal value. Therefore, the problem is not unbounded.

5 Introduction to linear programming

5.1 Linear functions and linear programming

Linear programming is the first class of optimization problems that we learn in this course. We will soon realize that the production planning example is a linear program. Here, the term "programming" sounds like some type of computer programming, but in fact, it is more like **planning**. Likewise, **mathematical programming** refers to classes of optimization problems that consist of an objective function and functional constraints. Then what does the other term "linear" mean? It means that the objective function and the constraints are *linear functions*, which we define next.

We say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is **linear** if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for any $x, y \in \mathbb{R}^d$ and $\alpha, \beta \in \mathbb{R}$. In fact, a function f is linear if and only if there exists some vector $c \in \mathbb{R}^d$ such that

$$f(x) = c^\top x = c_1 x_1 + c_2 x_2 + \cdots + c_d x_d = \sum_{j=1}^{d} c_j x_j.$$

Example 2.3. Let $x \in \mathbb{R}^3$. $x_1 + x_2 + x_3$, $x_1 + 2x_3$, x_2, and $-x_1 - x_2 + 3x_3$ are all linear functions.

Example 2.4. Let $x \in \mathbb{R}^3$. $x_1 x_3$ and $x_1^2 + 2x_1 x_2 + x_2^2$ are nonlinear.

A **linear program** (in short, **LP**) is an optimization of the following form.

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \leq b_i, \quad i \in [m], \\ & x \in \mathbb{R}^d \end{aligned} \tag{2.6}$$

where $c \in \mathbb{R}^d$ and $a_1, \ldots, a_m \in \mathbb{R}^d$. Here, the objective function $c^\top x$ is equal to

$$c_1 x_1 + c_2 x_2 + \cdots + c_d x_d,$$

and the constraint $a_i^\top x \leq b_i$ is equivalent to

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{id} x_d \leq b_i.$$

Example 2.5. Recall the production planning problem (2.4) where x and y are the variables for the numbers of products X and Y. The objective function is $x + y$, which is linear. Moreover, $2x + 3y$, $2x + y$, x, and y are all linear. Therefore, all constraints of (2.4) are given by linear functions. Thereofre, problem (2.4) is indeed a linear program.

5.2 Production planning

We have discussed the production planning problem of two types of products from two kinds of materials. Let us discuss a more general version of production planning.

Imagine a company producing a variety of products.

- There are d different products in the product portfolio of the company. We enumerate them by product 1, product 2, and so forth.
- There are m different kinds of materials necessary for producing the products. We enumerate them by material 1, material 2, and so forth.
- One unit of product j sells for price p_j for $i \in [d]$.
- The current stock of material i is b_i for $i \in [m]$.
- Producing one unit of project j requires a_{ij} amount of material i for every pair $(i, j) \in [m] \times [d]$. Here, $[m] \times [d]$ denote the set $\{(i, j) : i \in [m], j \in [d]\}$.
- We assume that all data are nonnegative, i.e. $p_j \geq 0$, $b_i \geq 0$, and $a_{ij} \geq 0$ for all $i \in [m]$ and $j \in [d]$.
- We assume that each product is divisible, which means that the amount of each product can be any real number. For example, we may produce 1.5 units of product 1.

Given this information, the goal is to decide the production quantity of each project under the current stock level of materials while maximizing the total revenue. Here, the revenue refers to the gross income by selling the products. As expected, we can resolve this question by writing a linear program (LP).

Remember the three components of a decision-making problem, decisions, constraints, and an objective.

1. Decisions: let x_j denote the amount of product j that we produce for $j \in [d]$.
2. Objective: the goal is to maximize the total revenue. Here, the total revenue is given by
$$\sum_{j\in[d]} p_j x_j.$$
3. Constraints: the total consumption of material i should be at most the current stock. The total consumption is
$$\sum_{j\in[d]} a_{ij} x_j.$$
As the current stock of material i is given by b_i, we can write the constraint as
$$\sum_{j\in[d]} a_{ij} x_j \leq b_i.$$
Moreover, we know that the production quantity of each product cannot be less than 0. Hence,
$$x \geq 0.$$
Here, $x \geq 0$ simply means that each component of x is nonnegative.

In summary, the production planning problem can be formulated as the following LP.

$$\begin{aligned} \max \quad & \sum_{j\in[d]} p_j x_j \\ \text{s.t.} \quad & \sum_{j\in[d]} a_{ij} x_j \leq b_i, \quad i \in [m], \\ & x \geq 0. \end{aligned}$$

Next let us consider a different objective.

- Suppose that one unit of material i incurs a cost of s_i for $i \in [m]$.

The profit is defined as the net income after deducting costs from earnings. The company may attempt to maximize the profit instead of the revenue. How can we model the new objective? Note that producing one unit of product j incurs a cost of

$$\sum_{i\in[m]} s_i a_{ij}.$$

Hence, the profit from selling one unit of product j is

$$p_j - \sum_{i\in[m]} s_i a_{ij}.$$

Then the new objective is to maximize

$$\sum_{j\in[d]} \left(p_j - \sum_{i\in[m]} s_i a_{ij} \right) x_j.$$

The corresonding LP is

$$\begin{aligned} \max \quad & \sum_{j \in [d]} \left(p_j - \sum_{i \in [m]} s_i a_{ij} \right) x_j \\ \text{s.t.} \quad & \sum_{j \in [d]} a_{ij} x_j \le b_i, \quad i \in [m], \\ & x \ge 0. \end{aligned}$$