IE 331 OR: Optimization **KAIST, Spring 2023**
Lecture #24: Two-period investment planning June 8, 2023
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1 Outline

In this lecture, we cover

- Errata in Value at Risk (VaR) materials (Lectures 19 and 20),
- two-period investment.

2 Errata: Value at Risk (VaR)

2.1 Lecture 19

Assume that we have likelikhood weights p_i for each scenario ξ_i and the distribution $\hat{P}_N$ with

$$\mathbb{P}_{\xi \sim \hat{P}_N}[\xi = \xi_i] = p_i, \quad i \in [N].$$

Fix some $\alpha \in (0,1)$. In Lecture 19, we defined the **Value-at-Risk** at level α or **α-VaR** is the risk measure defined as

$$\mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) = \min\left\{t : \mathbb{P}_{\xi\sim\hat{P}_N}[g(x,\xi) \le t] > \alpha\right\}.$$

There is a mistake in this definition. The correct definition is

$$\mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) = \min\left\{t : \mathbb{P}_{\xi\sim\hat{P}_N}[g(x,\xi) \le t] \ge \alpha\right\}$$

where the lower bound on the probability is given by a **non-strict inequality**. We also considered the following example.

Example 24.1. Suppose that we have

i	1	2	3	4	5	6
p_i	0.05	0.15	0.1	0.4	0.2	0.1
$g(x,\xi_i)$	10	8	6	3	2	-2
$\mathbb{P}_{\xi\sim\hat{P}_N}[g(x,\xi) \le g(x,\xi_i)]$	1	0.95	0.8	0.7	0.3	0.1

Then

- $\mathrm{VaR}_{0.98}\left(g(x,\xi); \hat{P}_N\right) = 10$.
- $\mathrm{VaR}_{0.95}\left(g(x,\xi); \hat{P}_N\right) = \not{10} \to 8$.
- $\mathrm{VaR}_{0.85}\left(g(x,\xi); \hat{P}_N\right) = 8$.
- $\mathrm{VaR}_{0.8}\left(g(x,\xi); \hat{P}_N\right) = \not{8} \to 6$.

- $\mathrm{VaR}_{0.7}\left(g(x,\xi);\hat{P}_N\right) = \emptyset \to 3$.
- $\mathrm{VaR}_{0.6}\left(g(x,\xi);\hat{P}_N\right) = 3$.

When $p_i = 1/N$ for $i \in [N]$ and $\alpha = 1-k/N$, then $\mathrm{VaR}_\alpha\left(g(x,\xi);\hat{P}_N\right)$ is **not the kth largest value but the $(k+1)$th largest value** among $g(x,\xi_1),\dots,g(x,\xi_N)$. Basically, if $g(x,\xi_1) \ge \cdots \ge g(x,\xi_N)$, then

$$\boldsymbol{\mathrm{VaR}_{1-k/N}\left(g(x,\xi);\hat{P}_N\right) = g(x,\xi_{k+1}), \quad k = 0,1,\dots.}$$

2.2 Lecture 20

Assume that we can model any constraint of the form $g(x,\xi_i) \le b_i$. Based on this, let us try to model

$$\mathrm{VaR}_\alpha\left(g(x,\xi);\hat{P}_N\right) \le 0.$$

This is equivalent to

$$\cancel{\min\left\{t:\ \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \le t\right] > \alpha\right\} \le 0} \to \min\left\{t:\ \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \le t\right] \ge \alpha\right\} \le 0.$$

We may rewrite this as

$$t \le 0$$

$$\cancel{\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \le t\right] > \alpha} \quad \to \quad \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \le t\right] \ge \alpha$$

Without loss of generality, we can take $t = 0$ and just consider

$$\cancel{\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \le 0\right] > \alpha} \quad \to \quad \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \le 0\right] \ge \alpha.$$

This is because $\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \le 0\right]$ never decreases as t increases.

Therefore, $\mathrm{VaR}_\alpha\left(g(x,\xi);\hat{P}_N\right) \le 0$ is equivalent to a **chance constraint**.

$$\cancel{\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \le 0\right] > \alpha} \quad \to \quad \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) \le 0\right] \ge \alpha \quad \Leftrightarrow \quad \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) > 0\right] \le 1-\alpha.$$

To model this, we introduce binary variables $z_i \in \{0,1\}$ for $i \in [N]$ for scenarios.

$$z_i = \begin{cases} 1, & \text{if } g(x,\xi_i) > 0 \\ 0, & \text{otherwise.} \end{cases}.$$

Basically, we add implications

$$z_i = 0 \quad \Rightarrow \quad g(x,\xi_i) \le 0, \quad i \in [N]$$

This can be modelled with the big-M technique:

$$g(x,\xi_i) \le M z_i, \quad i \in [N].$$

We need to ensure that the probability $g(x,\xi) > 0$ is no greater than $1-\alpha$:

$$\sum_{i\in[N]} p_i z_i \le 1-\alpha.$$

In summary,

$$\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & \mathrm{VaR}_\alpha \left(g(x,\xi); \hat{P}_N \right) \leq 0 \\
& x \in \mathcal{X}
\end{aligned}$$

is equivalent to

$$\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g(x,\xi_i) \leq M z_i, \quad i \in [N] \\
& \sum_{i \in [N]} p_i z_i \leq 1 - \alpha \\
& x \in \mathcal{X},\ z \in \{0,1\}^N
\end{aligned}$$

3 Two-period investment

Let us consider a two-period investment problem. Here, we have three stages of decisions in the optimization model. Remember that $r_{s,1}$ is the random return of stocks for period 1 and that $r_{s,2}$ is the random return for period 2. Suppose that there are n outcomes for period 1:

$$r_{s,1}^{(1)}, \ldots, r_{s,1}^{(n)}.$$

Next, under the ith outcome $r_{s,1}^{(i)}$ for period 1, we assume that n outcomes for period 2:

$$r_{s,1}^{(i,1)}, \ldots, r_{s,1}^{(i,n)}.$$

This is summarized as the following scenario tree. Since there are n outcomes for period 1 and n

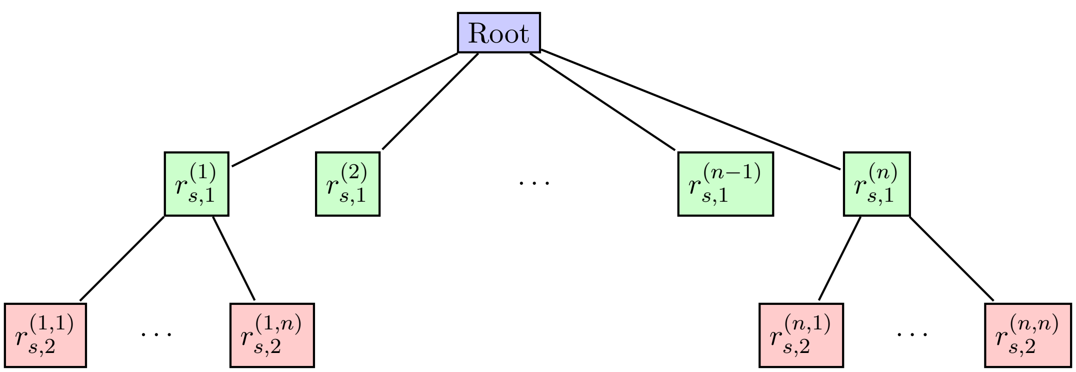

Figure 24.1: The scenario tree under n outcomes for each stage

outcomes for period 2, there are technically $n \times n = n^2$ scenarios. Moreover, we assume that each outcome occurs with equal probability $1/n$. More specifically,

$$\begin{aligned}
\mathbb{P}\left[r_{s,1} = r_{s,1}^{(i)} \right] &= \frac{1}{n}, \quad i = 1, \ldots, n \\
\mathbb{P}\left[r_{s,2} = r_{s,2}^{(i,j)} \mid r_{s,1} = r_{s,1}^{(i)} \right] &= \frac{1}{n}, \quad i = 1, \ldots, n,\ j = 1, \ldots, n
\end{aligned}$$

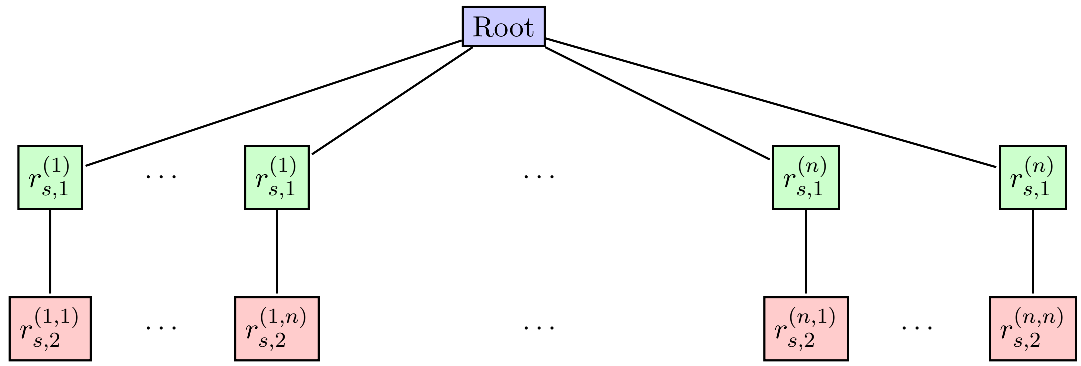

Figure 24.2: The scenario tree under n^2 scenarios

Decisions The period 1 investment decision is given by

$$x_1 = (x_{s,1}, x_{o,1})$$

where $x_{s,1}$ is for stocks and $x_{o,1}$ is for savings. At the end of period 1, we observe a value among

$$r_{s,1}^{(1)}, \ldots, r_{s,1}^{(n)},$$

each of which occurs with probability $1/n$. When the outcome is $r_{s,1}^{(i)}$, the period 2 investment decision is given by

$$x_2^{(i)} = (x_{s,2}^{(i)}, x_{o,2}^{(i)}), \quad i = 1, \ldots, n.$$

After period 2, we observe

$$r_{s,2}^{(i,1)}, \ldots, r_{s,2}^{(i,n)},$$

each of which occurs with probability $1/n$. When the outcome is $r_{s,2}^{(i,j)}$, the total wealth is

$$w_2^{(i,j)} = r_{s,2}^{(i,j)} x_{s,2}^{(i)} + x_{o,2}^{(i)}.$$

Third-stage model The third-stage is after period 2 where we collect the total reward from the two periods. Then the objective is to maximize the reward given by

$$\min\left\{p\left(w_2^{(i,j)} - G\right),\ q\left(w_2^{(i,j)} - G\right)\right\}$$

where p is the borrowing rate and q is the interest rate. Then the third-stage model is given by

$$\begin{aligned} \max \quad & \min\left\{p\left(w_2^{(i,j)} - G\right),\ q\left(w_2^{(i,j)} - G\right)\right\} \\ \text{s.t.} \quad & w_2^{(i,j)} = r_{s,2}^{(i,j)} x_{s,2}^{(i)} + x_{o,2}^{(i)}. \end{aligned}$$

We can represent this as the following linear program.

$$\begin{aligned} \max \quad & t^{(i,j)} \\ \text{s.t.} \quad & w_2^{(i,j)} = r_{s,2}^{(i,j)} x_{s,2}^{(i)} + x_{o,2}^{(i)} \\ & t^{(i,j)} \le p\left(w_2^{(i,j)} - G\right) \\ & t^{(i,j)} \le q\left(w_2^{(i,j)} - G\right) \end{aligned}$$

Second-stage model The second stage is after period 1 and before period 2. In the second stage, we prepare our second period investment plan. Assuming that the first period outcome is $r_{s,1}^{(i)}$, the wealth from period 1 would be

$$w_1^{(i)} = r_{s,1}^{(i)} x_{s,1} + x_{o,1}.$$

For period 2, we allocate the wealth to stocks and savings. Hence,

$$w_1^{(i)} = x_{s,2}^{(i)} + x_{o,2}^{(i)}.$$

Eliminating the term $w_1^{(i)}$, we can simply write

$$r_{s,1}^{(i)} x_{s,1} + x_{o,1} = x_{s,2}^{(i)} + x_{o,2}^{(i)}.$$

Moreover,

$$x_{s,2}^{(i)}, x_{o,2}^{(i)} \geq 0.$$

The objective is to maximize the expeted third-stage value

$$\frac{1}{n} \sum_{j=1}^{n} Q_3(x_1, x_2^{(i)}, r_{s,1}^{(i)}, r_{s,2}^{(i,j)}).$$

Then the second-stage model is given by

$$\begin{aligned} \max \quad & \frac{1}{n} \sum_{j=1}^{n} Q_3(x_1, x_2^{(i)}, r_{s,1}^{(i)}, r_{s,2}^{(i,j)}) \\ \text{s.t.} \quad & r_{s,1}^{(i)} x_{s,1} + x_{o,1} = x_{s,2}^{(i)} + x_{o,2}^{(i)} \\ & x_{s,2}^{(i)}, x_{o,2}^{(i)} \geq 0. \end{aligned}$$

First-stage model Note that the initial budget is B. Hence, we have

$$B = x_{s,1} + x_{o,1}.$$

Moreover, for simplicity, we assume no short selling and no leverage. Then

$$x_{s,1}, x_{o,1} \geq 0.$$

The first stage objective is to maximize the expeted second-stage value

$$\frac{1}{n} \sum_{i=1}^{n} Q_2(x_1, r_{s,1}^{(i)}).$$

Hence, the first-stage model is given by

$$\begin{aligned} \max \quad & \frac{1}{n} \sum_{i=1}^{n} Q_2(x_1, r_{s,1}^{(i)}) \\ \text{s.t.} \quad & x_{s,1} + x_{o,1} = B \\ & x_{s,1}, x_{o,1} \geq 0 \end{aligned}$$

Aggregated model The full model after aggregating the three stages is given as follows.

$$\begin{aligned}
\max \quad & \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} t^{(i,j)} \\
\text{s.t.} \quad & x_{s,1} + x_{o,1} = B \\
& x_{s,1}, x_{o,1} \geq 0 \\
& r_{s,1}^{(i)} x_{s,1} + x_{o,1} = x_{s,2}^{(i)} + x_{o,2}^{(i)}, \quad i = 1, \ldots, n \\
& x_{s,2}^{(i)}, x_{o,2}^{(i)} \geq 0, \quad i = 1, \ldots, n \\
& w_2^{(i,j)} = r_{s,2}^{(i,j)} x_{s,2}^{(i)} + x_{o,2}^{(i)}, \quad i = 1, \ldots, n, \ j = 1, \ldots, n \\
& t^{(i,j)} \leq p\left(w_2^{(i,j)} - G\right), \quad i = 1, \ldots, n, \ j = 1, \ldots, n \\
& t^{(i,j)} \leq q\left(w_2^{(i,j)} - G\right), \quad i = 1, \ldots, n, \ j = 1, \ldots, n
\end{aligned}$$