1 Outline

In this lecture, we cover

- introduction to multi-stage optimization,
- formulations based on scenario trees.

2 Introduction to multi-stage optimization

Many settings require us to make a sequence of decisions over time, as new information is revealed. This has been studied under several different names. To name a few,

- online learning and online algorithms,
- sequential decision-making,
- reinforcement learning,
- dynamic programming,
- multi-stage optimization.

Each has slightly different flavors. In this course, we study **multi-stage optimization**, where the focus is on how to model these problems using mathematical optimization. We will particularly focus on linear programming formulations of these problems.

3 Investment planning

We consider a simple investment example to illustrate the ideas. The base investment amount is B, and a target amount is G for a future expense (e.g., buying a property). Assume that we have T investment periods. Then the goal is to implement an investment plan so that the welath w_T at the end of period T can be used towards to expense. If $w_T > G$, then we **earn**

$$q(w_T - G)$$

where $q > 1$ is the interest rate, since we can deposit the excess $w_T - G$ into a bank account. If the final wealth $w_T \leq G$, then we **pay**

$$p(G - w_T)$$

where $p > q$ is the borrowing rate, as we need to borrrow $G - w_T$ to cover the expense. Succintly, the **reward** for generating wealth w_T is

$$\min\left\{p(w_T - G), q(w_T - G)\right\}.$$

There are two different instruments for investment, stocks (s) and savings (o). Let $t \in [T]$ be a period.

At the start of period t, we allocate amounts to stocks and savings, and let $x_{s,t}$ and $x_{o,t}$ denote these amounts, respectively. We have a simple spending constraint

$$\begin{aligned} w_{t-1} &= x_{s,t} + x_{o,t}, \quad t \geq 2 \\ w_0 &= B \end{aligned}$$

where w_{t-1} is the wealth generated from the previous period for $t \geq 2$. We can **short sell** stocks (borrow shares) hence $x_{s,t}$ can be negative, but we consider the setting where

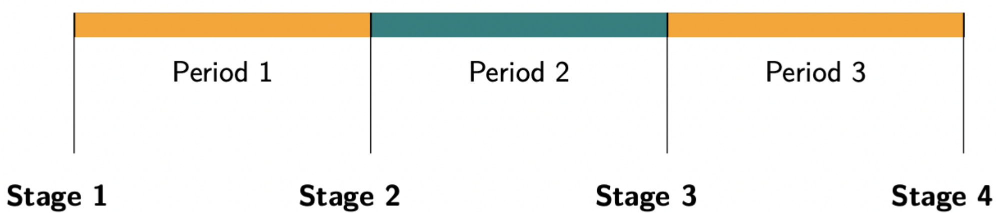
$$x_{s,t} \geq 0.$$

At the end of period t, for the amount saved $x_{o,t}$, we receive all of it back, with certainty. We receive returns on stocks wich are **random**, i.e., unknown at the start of the priod. Let $r_{s,t}$ be the respective return for stocks. Then the total wealth at the end of the period is

$$w_t = r_{s,t} x_{s,t} + x_{o,t}.$$

For simplicity, we assume no tax and no transaction fees, no discounting factors.

For three periods, there are four stages of decisions as follows.



- At stage 1 (the start of period 1), we allocate $x_1 = (x_{s,1}, x_{o,1})$.
- At stage 2 (the end of period 1/the start of period 2), we observe $r_{s,1}$, based on which we allocate $x_2 = (x_{s,2}, x_{o,2})$.
- At stage 3 (the end of period 2/the start of period 3), we observe $r_{s,2}$, based on which we allocate $x_3 = (x_{s,3}, x_{o,3})$.
- At stage 4 (the end of period 3), we observe $r_{s,3}$.

There are a couple of remarks. Some decisions must be made before information is revealed, while others can be made after. Our models need to account for the dependence between decisions made before information is revealed, and those made after.

4 Multi-stage optimization models

For general multi-stage models, the workflow is as follows. Consider period $t \in [T]$.

- We implement decision x_t for period t, **before observing** information ξ_t.
- Receive and observe information ξ_t.
- We prepare decision x_{t+1} for period $t+1$ based on $\{x_1, \dots, x_t\}$ and $\{\xi_1, \dots, \xi_t\}$.

In one picture, we have

$$\begin{aligned}
&\text{implement decision } x_1 \\
&\to \text{receive } \xi_1 \to \text{implement decision } x_2 \\
&\quad\vdots \\
&\to \text{receive } \xi_{T-2} \to \text{implement decision } x_{T-1} \\
&\to \text{receive } \xi_{T-1} \to \text{implement decision } x_T.
\end{aligned}$$

Here, we always start by implementing decision x_1 and finish by implementing decision x_T.

Let us define some notations for simpler presentation.

- $\xi = (\xi_1, \ldots, \xi_{T-1})$ is the vector of data across all stages.
- $x = (x_1, \ldots, x_T)$ is the vector of decisions across all stages.
- Here the components $\xi_1, \ldots, \xi_{T-1}$ and $x_1, \ldots, x_T$ themselves are also vectors, so think of ξ and x as stacked vectors.
- For $t \in [T]$, $\xi_{[t]} = (\xi_1, \ldots, \xi_t)$ and $x_{[t]} = (x_1, \ldots, x_t)$ are vectors representing data and decisions from stages 1 to t, inclusive.

Define the final-stage decision model as follows.

$$\begin{aligned}
Q_T(x_{[T-1]}, \xi_{[T-1]}) \quad := \quad \min_{x_T} \quad & c_T(\xi_{[T-1]})^\top x_T \\
\text{s.t.} \quad & A_T(\xi_{[T-1]}) x_{[T-1]} + B_T(\xi_{[T-1]}) x_T \geq b_T(\xi_{[T-1]}).
\end{aligned}$$

Here, $c_T(\xi_{[T-1]})$ is the cost vector for stage T determined by the past data $\xi_{[T-1]}$, and x_T is the decision made at the final-stage. $A_T(\xi_{[T-1]})$ and $B_T(\xi_{[T-1]})$ are the constraint matrices for stage T determined by the past data $\xi_{[T-1]}$.

Define the stage-t decision model for $2 \leq t \leq T-1$ as follows.

$$\begin{aligned}
Q_t(x_{[t-1]}, \xi_{[t-1]}) \quad := \quad \min_{x_t} \quad & c_t(\xi_{[t-1]})^\top x_t + \mathbb{E}\left[Q_{t+1}(x_{[t]}, \xi_{[t]}) \mid \xi_{[t-1]}\right] \\
\text{s.t.} \quad & A_t(\xi_{[t-1]}) x_{[t-1]} + B_t(\xi_{[t-1]}) x_t \geq b_t(\xi_{[t-1]}).
\end{aligned}$$

Again, $c_t(\xi_{[t-1]})$ is the cost vector for stage t determined by the past data $\xi_{[t-1]}$, and x_t is the decision made at stage t. $A_t(\xi_{[t-1]})$ and $B_t(\xi_{[t-1]})$ are the constraint matrices for stage t determined by the past data $\xi_{[t-1]}$. Moreover,

$$\mathbb{E}\left[Q_{t+1}(x_{[t]}, \xi_{[t]}) \mid \xi_{[t-1]}\right]$$

is the expectation of the stage-$(t+1)$ value $Q_{t+1}(x_{[t]}, \xi_{[t]})$ conditional on the data $\xi_{[t-1]}$ up to stage $t-1$. Basically, when computing the conditional expectation, we consider the randomness of ξ_t only.

Lastly, we define the first-stage decision model as

$$\begin{aligned}
\min_{x_1} \quad & c_1^\top x_1 + \mathbb{E}\left[Q_2(x_1, \xi_1)\right] \\
\text{s.t.} \quad & B_1 x_1 \geq b_1.
\end{aligned}$$

Hence, when we make decision x_1, we consider $Q_2(x_1, \xi_1)$. In fact, $Q_2(x_1, \xi_1)$ is determined by the data across not only the current stage ξ_1 and but also all future stages $\xi_2, \ldots, \xi_{T-1}$. Although we make decision for stage 1, we need to look ahead future stages based on predictions about the future.

5 Formulation based on scenario tree

The two-stage optimization model is given by

$$\begin{aligned} \min_{x_1} \quad & c_1^\top x_1 + \mathbb{E}\left[Q_2(x_1, \xi_1)\right] \\ \text{s.t.} \quad & B_1 x_1 \geq b_1 \end{aligned}$$

where

$$\begin{aligned} Q_2(x_1, \xi_1) \quad := \quad \min_{x_2} \quad & c_2(\xi_1)^\top x_2 \\ \text{s.t.} \quad & A_2(\xi_1) x_1 + B_2(\xi_1) x_2 \geq b_2(\xi_1). \end{aligned}$$

To solve this, we used **scenarios** to predict the value of ξ_1. For more than two stages, we also need to consider $\xi_2, \ldots, \xi_{T-1}$. Hence, we use what is called a **scenario tree** since the information is revealed sequentially, and we need to account for the information structure. Let $\xi^1, \ldots, \xi^N$ be our scenarios. Here,

$$\xi^i = (\xi_1^i, \ldots, \xi_{T-1}^i)$$

and ξ_t^i is the data observed at the end of stage $t \in [T-1]$.

For example, let us consider the case with $T = 3$, i.e., a three-stage model. Note that the three-stage model is given by

$$\begin{aligned} \min_{x_1} \quad & c_1^\top x_1 + \mathbb{E}\left[Q_2(x_1, \xi_1)\right] \\ \text{s.t.} \quad & B_1 x_1 \geq b_1 \end{aligned}$$

where

$$\begin{aligned} Q_2(x_1, \xi_1) \quad := \quad \min_{x_2} \quad & c_2(\xi_1)^\top x_2 + \mathbb{E}\left[Q_3((x_1, x_2), (\xi_1, \xi_2)) \mid \xi_1\right] \\ \text{s.t.} \quad & A_2(\xi_1) x_1 + B_2(\xi_1) x_2 \geq b_2(\xi_1) \end{aligned}$$

and

$$\begin{aligned} Q_3((x_1, x_2), (\xi_1, \xi_2)) \quad := \quad \min_{x_3} \quad & c_3(\xi_1, \xi_2)^\top x_3 \\ \text{s.t.} \quad & A_3(\xi_1, \xi_2)(x_1, x_2) + B_3(\xi_1, \xi_2) x_3 \geq b_3(\xi_1, \xi_2). \end{aligned}$$

Suppose that we five scenarios.

$$\begin{aligned} \xi^1 &= (\xi_1^1, \xi_2^1) \\ \xi^2 &= (\xi_1^2, \xi_2^2) \\ \xi^3 &= (\xi_1^3, \xi_2^3) \\ \xi^4 &= (\xi_1^4, \xi_2^4) \\ \xi^5 &= (\xi_1^5, \xi_2^5). \end{aligned}$$

We represent multiple stages within scenarios as the tree structure as in Figure 23.1. This is a scenario tree. Assume that scenarios $\xi_1, \ldots, \xi_5$ occur with equal probability:

$$\mathbb{P}\left[\xi = \xi^1\right] = \cdots = \mathbb{P}\left[\xi = \xi^5\right] = \frac{1}{5}.$$

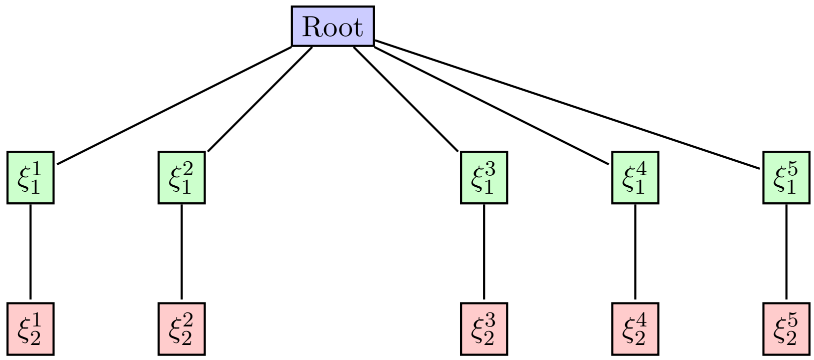


Figure 23.1: A scenario tree for $T = 3$

Then

$$\mathbb{E}\left[Q_2(x_1, \xi_1)\right] = \sum_{i \in [5]} \mathbb{P}\left[\xi = \xi^i\right] \cdot Q_2(x_1, \xi_1^i) = \frac{1}{5} \sum_{i \in [5]} Q_2(x_1, \xi_1^i).$$

Moreover, by the scenario tree structure, we know that if the first-stage information is ξ_1^i, then the second-stage outcome would be ξ_2^i. Therefore,

$$\mathbb{E}\left[Q_3((x_1, x_2), (\xi_1, \xi_2)) \mid \xi_1 = \xi_1^i\right] = Q_3((x_1, x_2), (\xi_1^i, \xi_2^i)).$$

For each scenario $i \in [5]$, we define variable

$$(x_1, x_2^i, x_3^i)$$

for three stages. Then the resulting optimization model is given by

$$\begin{aligned} \min_{x_1} \quad & c_1^\top x_1 + \frac{1}{5} \sum_{i \in [5]} Q_2(x_1, \xi_1^i) \\ \text{s.t.} \quad & B_1 x_1 \geq b_1 \end{aligned}$$

where

$$\begin{aligned} Q_2(x_1, \xi_1^i) \quad := \quad \min_{x_2^i} \quad & c_2(\xi_1^i)^\top x_2^i + Q_3((x_1, x_2^i), (\xi_1^i, \xi_2^i)) \\ \text{s.t.} \quad & A_2(\xi_1^i) x_1 + B_2(\xi_1^i) x_2^i \geq b_2(\xi_1^i) \end{aligned}$$

and

$$\begin{aligned} Q_3((x_1, x_2^i), (\xi_1^i, \xi_2^i)) \quad := \quad \min_{x_3^i} \quad & c_3(\xi_1^i, \xi_2^i)^\top x_3^i \\ \text{s.t.} \quad & A_3(\xi_1^i, \xi_2^i)(x_1, x_2^i) + B_3(\xi_1^i, \xi_2^i) x_3^i \geq b_3(\xi_1^i, \xi_2^i). \end{aligned}$$

Here, we use variables x_2^i, x_3^i for scenario i. Therefore, the aggregated model is given by

$$\begin{aligned} \min \quad & c_1^\top x_1 + \frac{1}{5} \sum_{i \in [5]} \left(c_2(\xi_1^i)^\top x_2^i + c_3(\xi_1^i, \xi_2^i)^\top x_3^i \right) \\ \text{s.t.} \quad & B_1 x_1 \geq b_1 \\ & A_2(\xi_1^i) x_1 + B_2(\xi_1^i) x_2^i \geq b_2(\xi_1^i), \quad i \in [5] \\ & A_3(\xi_1^i, \xi_2^i)(x_1, x_2^i) + B_3(\xi_1^i, \xi_2^i) x_3^i \geq b_3(\xi_1^i, \xi_2^i), \quad i \in [5]. \end{aligned}$$

6 Aggregating scenario trees

What if we only have **two possible outcomes** for ξ_1 (first stage data). Let ξ_1^a and ξ_1^b denote the two outcomes. Suppose that

- $\xi_1^1 = \xi_1^2 = \xi_1^a$,
- $\xi_1^3 = \xi_1^4 = \xi_1^5 = \xi_1^b$.

Here, if we see ξ_1^a, we **cannot tell** if we are in scenario 1 or 2. If we see ξ_1^b, we **cannot tell** if we are in scenario 3, 4, or 5. Figure 23.2 represents the scenario tree for this case. Note that we

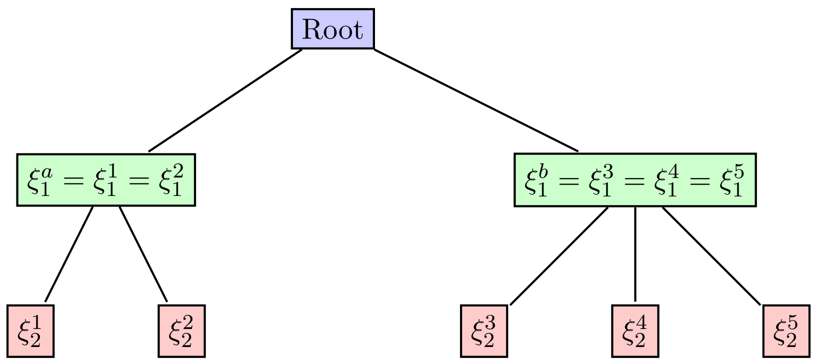


Figure 23.2: The scenario tree aggregating the same second stage outcomes

may have $\xi_2^1 = \xi_2^3$. However, we use distinct nodes for ξ_2^1 and ξ_2^3, because their parent nodes are different. We use the notion of **non-anticipative constraints** derived by examining the scenario tree given in Figure 23.2. We add the following constraints according to the scenario tree given in Figure 23.2.

$$\begin{aligned} x_2^1 &= x_2^2 \quad \text{(level 1, left node)} \\ x_2^3 = x_2^4 &= x_2^5 \quad \text{(level 1, right node)} \end{aligned}$$

Then the resulting optimization model is given by

$$\begin{aligned} \min \quad & c_1^\top x_1 + \frac{1}{5}\sum_{i\in[5]} \left(c_2(\xi_1^i)^\top x_2^i + c_3(\xi_1^i,\xi_2^i)^\top x_3^i \right) \\ \text{s.t.} \quad & B_1 x_1 \geq b_1 \\ & A_2(\xi_1^i)x_1 + B_2(\xi_1^i)x_2^i \geq b_2(\xi_1^i), \quad i\in[5] \\ & A_3(\xi_1^i,\xi_2^i)(x_1,x_2^i) + B_3(\xi_1^i,\xi_2^i)x_3^i \geq b_3(\xi_1^i,\xi_2^i), \quad i\in[5] \\ & x_2^1 = x_2^2 \\ & x_2^3 = x_2^4 = x_2^5. \end{aligned}$$

We can get a more compact representation by replacing each non-anticipative constraint with a single variable, according to the scenario tree as in Figure 23.3. Basically, we replace both x_2^1 and x_2^2 by x_2^a, and we replace x_2^3, x_2^4, x_2^5 by x_2^b. Then we deduce the following optimization model.

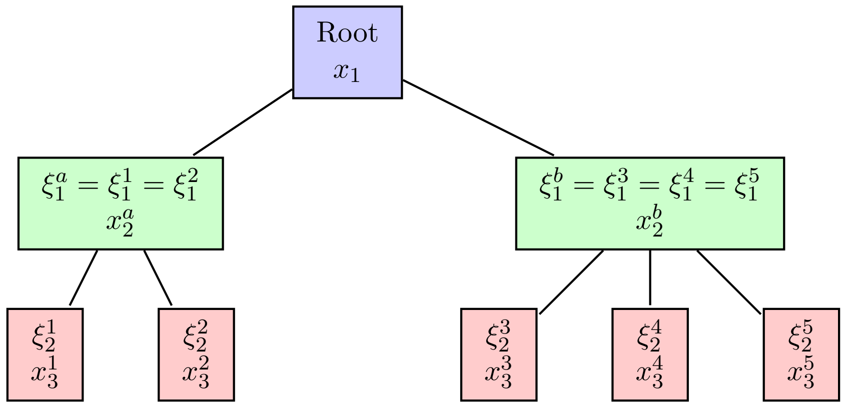


Figure 23.3: The scenario tree with corresponding variables

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\begin{aligned}
\min \quad & c_1^\top x_1 + \left(\frac{2}{5} c_2(\xi_1^a) x_2^a + \frac{3}{5} c_2(\xi_1^b) x_2^b\right) + \frac{1}{5} c_3(\xi_1^a, \xi_2^1)^\top x_3^1 + \frac{1}{5} c_3(\xi_1^a, \xi_2^2)^\top x_3^2 \\
& \quad + \frac{1}{5} c_3(\xi_1^b, \xi_2^3)^\top x_3^3 + \frac{1}{5} c_3(\xi_1^b, \xi_2^4)^\top x_3^4 + \frac{1}{5} c_3(\xi_1^b, \xi_2^5)^\top x_3^5 \\
\text{s.t.} \quad & B_1 x_1 \geq b_1 \\
& A_2(\xi_1^a) x_1 + B_2(\xi_1^a) x_2^a \geq b_2(\xi_1^a) \\
& A_2(\xi_1^b) x_1 + B_2(\xi_1^b) x_2^b \geq b_2(\xi_1^b) \\
& A_3(\xi_1^a, \xi_2^1)(x_1, x_2^a) + B_3(\xi_1^a, \xi_2^1) x_3^1 \geq b_3(\xi_1^a, \xi_2^1) \\
& A_3(\xi_1^a, \xi_2^2)(x_1, x_2^a) + B_3(\xi_1^a, \xi_2^2) x_3^2 \geq b_3(\xi_1^a, \xi_2^2) \\
& A_3(\xi_1^b, \xi_2^3)(x_1, x_2^b) + B_3(\xi_1^b, \xi_2^3) x_3^3 \geq b_3(\xi_1^b, \xi_2^3) \\
& A_3(\xi_1^b, \xi_2^4)(x_1, x_2^b) + B_3(\xi_1^b, \xi_2^4) x_3^4 \geq b_3(\xi_1^b, \xi_2^4) \\
& A_3(\xi_1^b, \xi_2^5)(x_1, x_2^b) + B_3(\xi_1^b, \xi_2^5) x_3^5 \geq b_3(\xi_1^b, \xi_2^5).
\end{aligned}
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