1 Outline

In this lecture, we cover

- the newsvendor problem with various risk measures,
- two-stage facility location.

2 Modeling newsvendor problem with Value at Risk

Again, we may reformulate the problem $\max_x f(x,\xi)$ as

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & f(x,\xi) \geq v. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & v - f(x,\xi) \leq 0. \end{aligned}$$

Then the corresponding value at risk is given by

$$\mathrm{VaR}_\alpha\left(v - f(x,\xi); \hat{P}_N\right) = \min\left\{t : \ \mathbb{P}_{\xi\sim\hat{P}_N}\left[v - f(x,\xi) \leq t\right] > \alpha\right\}.$$

We learned that

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & \mathrm{VaR}_\alpha\left(v - f(x,\xi); \hat{P}_N\right) \leq 0 \end{aligned}$$

can be reformulated as

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & v - f(x,\xi_i) \leq M z_i, \quad i \in [N] \\ & \sum_{i\in[N]} p_i z_i \leq 1 - \alpha \\ & x \in \mathbb{Z}_+, \ z \in \{0,1\}^N. \end{aligned}$$

Plugging in the formula $f(x,\xi) = \min\{(p+h)\xi - (c+h)x, \ (p-c)x\}$, we obtain

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & v - M z_i \leq (p+h)\xi - (c+h)x, \quad i \in [N] \\ & v - M z_i \leq (p-c)x, \quad i \in [N] \\ & \sum_{i\in[N]} p_i z_i \leq 1 - \alpha \\ & x \in \mathbb{Z}_+, \ z \in \{0,1\}^N. \end{aligned}$$

3 Two-stage facility location

We revisit the facility location problem again. In particular, we consider and model the problem in a pandemic situation. In a pandemic setting, testing the population for disease is key to managing its spread in a city. Basically, we want to determine locations for testing facilities.

Let each suburb be indexed by $j \in [n]$. The number of people that need testing on a given day is ξ^j for $j \in [n]$. We must decide where to place a limited number of **testing centers**, which provide testing services. Distances between suburbs j and k are given by d_{jk}. Travel is not recommended during a pandemic, so we want to place testing centers strategically to minimize travel.

In fact, we have an option of placing **mobile testing centers** at certain locations. Here, the mobile testing centers can move to locations of high demand if needed, in order to further reduce travel. However, these have limited capacity c.

Tesing numbers, the nubmers of people tested on a day, are **uncertain**, but we have estimates in the form of **scenarios**. Basically, we assume that data

$$\xi_1^j, \ldots, \xi_N^j$$

recording the numbers of test cases for the past N days from suburb j.

Testing centers need to be decided

here-and-now

because they take a long time to set up. They may involve machinery to process the tests. Mobile testing centers are quick to set up, thus we can potentially employ a

wait-and-see

approach. What time means is that we wait until we know the exact testing demand (or at least a better estimate of it) before we decide where to set up the mobile testing centers.

The question is, how can we test the population with least travel? We want to minimize the total expected distance travelled.

3.1 Warm-up: deterministic demand setting

To help up formulate the model with uncertainty in demand, let us try to understand the case when we have **no uncertainty**. Basically, we know the value of ξ^j for each $j \in [n]$ exactly.

Decisions. We use binary variable $x_j \in \{0, 1\}$ to indicate whether to place a testing center at location $j \in [n]$. Similarly, we use binary variable $y_j \in \{0, 1\}$ to describe whether we build a mobile testing center at $j \in [n]$.

Moreover, we need **allocation variables** that tell us where people are going to get tested. For $j, k \in [n]$, we use variable z_{jk} to capture the number of people from location j that will get tested at a testing center or a mobile center at location k.

Objective. We minimize total travel distance. Hence, we consider

$$\min \quad \sum_{j \in [n]} \sum_{k \in [n]} d_{jk} z_{jk}.$$

Constraints. First, we assume that $x_j, y_j \in \{0,1\}$ and $z_{jk} \geq 0$ for $j,k \in [n]$.

Next, we assume that everyone in the population must be tested.

$$\sum_{k\in[n]} z_{jk} = \xi^j, \quad j \in [n].$$

We can build at most K testing centers and at most L mobile centers.

$$\sum_{j\in[n]} x_j \leq K, \quad \sum_{j\in[n]} y_j \leq L.$$

Most importantly, we cannot send people to a location where there is no testing center. Moreover, a mobile testing center has a capacity of c on the number of tests. We can capture theses by a single constraint

$$\sum_{j\in[n]} z_{jk} \leq M_k x_k + c y_k, \quad k \in [n]$$

where M_k is a big-M constant. Note that

$$\sum_{j\in[n]} z_{jk} \leq \sum_{j\in[n]}\sum_{k\in[n]} z_{jk} \leq \sum_{j\in[n]} \xi^j,$$

and therefore, we may set M_k as

$$M_k = \sum_{j\in[n]} \xi^j.$$

Therefore, the resulting optimization model is given by

$$\begin{aligned}
\min \quad & \sum_{j\in[n]}\sum_{k\in[n]} d_{jk} z_{jk} \\
\text{s.t.} \quad & \sum_{k\in[n]} z_{jk} = \xi^j, \quad j \in [n] \\
& \sum_{j\in[n]} x_j \leq K, \\
& \sum_{j\in[n]} y_j \leq L, \\
& \sum_{j\in[n]} z_{jk} \leq M_k x_k + c y_k, \quad k \in [n] \\
& x \in \{0,1\}^n,\ y \in \{0,1\}^n,\ z \geq 0.
\end{aligned}$$

3.2 Uncertain demand case

Now we consider the case when we do not have the testing numbers $\xi^1, \ldots, \xi^n$ exactly, but we have access to scenarios

$$\xi^j_1, \ldots, \xi^j_N, \quad j \in [n].$$

One approach is to consider individual scenarios separately. For each scenario $i \in [N]$, we solve

$$\begin{aligned}
\min \quad & \sum_{j\in[n]}\sum_{k\in[n]} d_{jk} z_{jk}^i \\
\text{s.t.} \quad & \sum_{k\in[n]} z_{jk}^i = \xi_i^j, \quad j \in [n] \\
& \sum_{j\in[n]} x_j^i \leq K, \\
& \sum_{j\in[n]} y_j^i \leq L, \\
& \sum_{j\in[n]} z_{jk}^i \leq M_k x_k^i + c y_k^i, \quad k \in [n] \\
& x^i \in \{0,1\}^n,\ y^i \in \{0,1\}^n,\ z^i \geq 0.
\end{aligned}$$

Let (x^i, y^i, z^i) be an optimal solution under scenario $i \in [N]$. Given the N solutions for N different scenarios, how do we decide the locations of testing centers for the future? We may take their average.

$$\begin{aligned}
\min \quad & \frac{1}{N}\sum_{i\in[N]}\sum_{j\in[n]}\sum_{k\in[n]} d_{jk} z_{jk}^i \\
\text{s.t.} \quad & \sum_{k\in[n]} z_{jk}^i = \xi_i^j, \quad j \in [n],\ i \in [N] \\
& \sum_{j\in[n]} x_j^i \leq K, \quad i \in [N] \\
& \sum_{j\in[n]} y_j^i \leq L, \quad i \in [N] \\
& \sum_{j\in[n]} z_{jk}^i \leq M_k x_k^i + c y_k^i, \quad k \in [n],\ i \in [N] \\
& x^i \in \{0,1\}^n,\ y^i \in \{0,1\}^n,\ z^i \geq 0, \quad i \in [N].
\end{aligned}$$

However, it is not too trivial to determine the locations, because different scenarios suggest different sets of locations. How do we aggregate them? More precisely, how do we choose a binary vector $x \in \{0,1\}^n$ based on N binary vectors $x^1, \ldots, x^N \in \{0,1\}^n$? Their average is not necessarily a binary vector, i.e.,

$$\frac{1}{N}\sum_{i\in[N]} x^i \notin \{0,1\}^n.$$

Another approach is to use some risk measure. Based on the N scenarios, we predict that the demand from location j is

$$\rho\left(\left\{\xi_i^j : i \in [N]\right\}\right).$$

Then we solve

$$
\begin{aligned}
\min \quad & \sum_{j\in[n]} \sum_{k\in[n]} d_{jk} z_{jk} \\
\text{s.t.} \quad & \sum_{k\in[n]} z_{jk} = \rho\left(\left\{\xi_i^j : i \in [N]\right\}\right), \quad j \in [n] \\
& \sum_{j\in[n]} x_j \le K, \\
& \sum_{j\in[n]} y_j \le L, \\
& \sum_{j\in[n]} z_{jk} \le M_k x_k + c y_k, \quad k \in [n] \\
& x \in \{0,1\}^n,\ y \in \{0,1\}^n,\ z \ge 0.
\end{aligned}
$$

Depending on the prediction quality of the risk measure, this can be a reasonable proxy for the formulation under the full knowledge of demands. However, this ignores the flexibility of opening mobile testing centers since we decide the locations of mobile centers before seeing the testing demand.

Motivated by this, we take the so-called **two-stage approach**. The basic idea is as follows.

1. Decide on x, the locations of testing centers, **before** seeing the testing demand (using the scenarios as information).
2. Then decide on y, the locations of mobile testing centers, **after** seeing the demand.

To see how to model the two-stage approach, we review the formulation given by

$$
\begin{aligned}
\min \quad & \frac{1}{N} \sum_{i\in[N]} \sum_{j\in[n]} \sum_{k\in[n]} d_{jk} z_{jk}^i \\
\text{s.t.} \quad & \sum_{k\in[n]} z_{jk}^i = \xi_i^j, \quad j \in [n] \\
& \sum_{j\in[n]} x_j^i \le K, \\
& \sum_{j\in[n]} y_j^i \le L, \\
& \sum_{j\in[n]} z_{jk}^i \le M_k x_k^i + c y_k^i, \quad k \in [n] \\
& x^i \in \{0,1\}^n,\ y^i \in \{0,1\}^n,\ z^i \ge 0, \quad i \in [N].
\end{aligned}
$$

Note that we decide x before seeing the demand. What this means is that we have the same x for all scenarios. Hence,

$$x^1 = \cdots = x^N = x.$$

Plugging in this to the formaultion, we deduce

$$
\begin{aligned}
\min \quad & \frac{1}{N} \sum_{i \in [N]} \sum_{j \in [n]} \sum_{k \in [n]} d_{jk} z^i_{jk} \\
\text{s.t.} \quad & \sum_{k \in [n]} z^i_{jk} = \xi^j_i, \quad j \in [n] \\
& \sum_{j \in [n]} x_j \leq K, \\
& \sum_{j \in [n]} y^i_j \leq L, \\
& \sum_{j \in [n]} z^i_{jk} \leq M_k x_k + c y^i_k, \quad k \in [n] \\
& x \in \{0,1\}^n,\ y^i \in \{0,1\}^n,\ z^i \geq 0, \quad i \in [N].
\end{aligned}
$$

Here x is called the **first-stage variable**, and y is called the **second-stage variable**.