**IE 331 OR: Optimization** **KAIST, Spring 2023**
**Lecture #20: Newsvendor problem with various risk measures** May 11, 2023
Lecturer: Dabeen Lee

# 1 Outline

In this lecture, we cover

- modeling with VaR,
- comparing risk measures,
- the newsvendor problem with various risk measures.

# 2 Modeling with VaR

Assume that we can model any constraint of the form $g(x, \xi_i) \leq b_i$. Based on this, let us try to model

$$\mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) \leq 0.$$

This is equivalent to

$$\cancel{\min\left\{t:\ \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\leq t\right] > \alpha\right\} \leq 0} \to \min\left\{t:\ \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\leq t\right] \geq \alpha\right\} \leq 0.$$

Remember that

$$\mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) = \min\left\{t:\ \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\leq t\right] \geq \alpha\right\}.$$

Then we may rewrite this as

$$\begin{aligned} & t \leq 0 \\ \cancel{\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\leq t\right] > \alpha} \quad \to \quad & \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\leq t\right] \geq \alpha \end{aligned}$$

Without loss of generality, we can take $t = 0$ and just consider

$$\cancel{\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\leq 0\right] > \alpha} \quad \to \quad \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\leq 0\right] \geq \alpha.$$

This is because $\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\leq 0\right]$ never decreases as $t$ increases.

Therefore, $\mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) \leq 0$ is equivalent to a **chance constraint**.

$$\cancel{\mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\leq 0\right] > \alpha} \quad \to \quad \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\leq 0\right] \geq \alpha \quad \Leftrightarrow \quad \mathbb{P}_{\xi\sim\hat{P}_N}\left[g(x,\xi) > 0\right] \leq 1-\alpha.$$

To model this, we introduce binary variables $z_i \in \{0,1\}$ for $i \in [N]$ for scenarios.

$$z_i = \begin{cases} 1, & \text{if } g(x,\xi_i) > 0 \\ 0, & \text{otherwise.} \end{cases}.$$

Basically, we add implications

$$z_i = 0 \quad \Rightarrow \quad g(x,\xi_i) \leq 0, \quad i \in [N]$$

This can be modelled with the big-M technique:

$$g(x,\xi_i) \le Mz_i, \quad i \in [N].$$

We need to ensure that the probability $g(x,\xi) > 0$ is no greater than $1-\alpha$:

$$\sum_{i\in[N]} p_i z_i \le 1-\alpha.$$

In summary,

$$\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & \mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) \le 0 \\
& x \in \mathcal{X}
\end{aligned}$$

is equivalent to

$$\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g(x,\xi_i) \le Mz_i, \quad i \in [N] \\
& \sum_{i\in[N]} p_i z_i \le 1-\alpha \\
& x \in \mathcal{X},\ z \in \{0,1\}^N
\end{aligned}$$

# 3 Which risk measure should we use?

## 3.1 Modeling aspects

Value at Risk (VaR) and Conditional Value at Risk (CVaR) can adjust the **degree of risk aversion** by adjusting the parameter $\alpha$. The higher the value of $\alpha$, the greater the risk aversion of the decision $x$. Note that

$$\begin{aligned}
\lim_{\alpha\to 1} \mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) &= \max_{i\in[N]} g(x,\xi_i) \\
\lim_{\alpha\to 1} \mathrm{CVaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) &= \max_{i\in[N]} g(x,\xi_i).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{\alpha\to 1} \mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) \le 0 \quad &\Leftrightarrow \quad g(x,\xi_i) \le 0 \quad \forall i \in [N] \\
\lim_{\alpha\to 1} \mathrm{CVaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) \le 0 \quad &\Leftrightarrow \quad g(x,\xi_i) \le 0 \quad \forall i \in [N]
\end{aligned}$$

In contrast,

$$\begin{aligned}
\lim_{\alpha\to 0} \mathrm{VaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) &= \min_{i\in[N]} g(x,\xi_i) \\
\lim_{\alpha\to 0} \mathrm{CVaR}_\alpha\left(g(x,\xi); \hat{P}_N\right) &= \mathbb{E}_{\xi\sim\hat{P}_N}\left[g(x,\xi)\right].
\end{aligned}$$

Moreover, we always have

$$\mathrm{CVaR}_\alpha\left(g(x,\xi);\hat{P}_N\right) \geq \mathrm{VaR}_\alpha\left(g(x,\xi);\hat{P}_N\right).$$

The Conditional Value at Risk takes into account the **tail risk/tail probability** whereas the Value at Risk does not. Considering the tail risk is important for some applications, but it leads to more conservative solutions.

A guideline is as follows. CVaR is used when we have uncertain objectives or when computing VaR is too demanding. VaR is used when we have uncertain constraints.

### 3.2 Computational aspects

Expectation and worst-case value are relatively simple. Computing expectation requires a new function

$$\frac{1}{N}\sum_{i=1}^{N} g(x,\xi_i),$$

which is essentially the sum of functions $g(x,\xi_1),\ldots,g(x,\xi_N)$. The worst-case vlaue requires $N$ constraints

$$g(x,\xi_i) \leq 0 \quad i \in [N].$$

CVaR requires adding new variables $t, r_1, \ldots, r_N$ that are all continuous and imposing $2N+1$ new constraints given by

$$\begin{aligned} t + \frac{1}{1-\alpha}\sum_{i=1}^{N} p_i r_i \leq 0 \\ r_i \geq 0 \quad \forall i \in [N] \\ t + r_i \geq g(x,\xi_i) \quad \forall i \in [N] \end{aligned}$$

VaR requires adding new binary variables $z_1, \ldots, z_N$ and $N+1$ new constraints given by

$$\begin{aligned} g(x,\xi_i) &\leq M z_i, \quad i \in [N] \\ \sum_{i\in[N]} p_i z_i &\leq 1-\alpha \end{aligned}$$

Moreover, when using VaR and CVaR, we need to choose the parameter $\alpha$.

## 4 Newsvendor problem

Recall the newsvendor problem. The newsvendor must choose a quantity $x \in \mathbb{Z}_+$ of newpapers to print on a particular day. The cost of printing a copy is $c > 0$. Each copy is sold at price $p$, and we assume that $p > c$. We also assume that each copy of unsold newspapers must be discarded at cost $h > 0$. The profit from printing $x$ copies under customer demand $\xi$ is given by

$$f(x,\xi) = p \cdot \min\{x,\xi\} - cx - h \cdot \max\{0, x-\xi\}.$$

The demand $\xi$ is unknown, but we have historical data $\xi_1, \ldots, \xi_N$ that can be treated as scenarios. Note that

$$f(x,\xi) = \begin{cases} p\xi - cx - h(x-\xi), & \text{if } x \geq \xi, \\ px - cx & \text{if } x < \xi. \end{cases}$$

Moreover, $p\xi - cx - h(x-\xi) \leq px - cx$ if and only if $x \geq \xi$. Hence, it follows that

$$f(x,\xi) = \min\{(p+h)\xi - (c+h)x,\ (p-c)x\}.$$

Here, we want to **maximize** the profit. Although the demand $\xi$ is unknown, we define risk measures based on the data set $\{\xi_1, \ldots, \xi_N\}$. Consider

$$\rho_f\left(x; \{\xi_i\}_{i\in[N]}\right).$$

Then we want to solve

$$\max_x \quad \rho_f\left(x; \{\xi_i\}_{i\in[N]}\right).$$

Previously, we considered risk measures for the constraint function, with the goal of achieving a low value. However, for the newsvendor problem, we want to achieve a high profit. Nevertheless, the developments for the constraint function extend to the maximizing objective.

### 4.1 Expectation

The expectation risk measure is given by

$$\rho_f\left(x; \{\xi_i\}_{i\in[N]}\right) = \frac{1}{N}\sum_{i=1}^N f(x,\xi_i).$$

Basically, we want to solve

$$\max \quad \frac{1}{N}\sum_{i=1}^N f(x,\xi_i).$$

This can be reformulated as

$$\begin{aligned} \max \quad & \frac{1}{N}\sum_{i=1}^N t_i \\ \text{s.t.} \quad & f(x,\xi_i) \geq t_i, \quad \forall i \in [N] \\ & x \in \mathbb{Z}_+ \end{aligned}$$

Plugging in the formula $f(x,\xi) = \min\{(p+h)\xi - (c+h)x,\ (p-c)x\}$, we obtain

$$\begin{aligned} \min \quad & \frac{1}{N}\sum_{i=1}^N t_i \\ \text{s.t.} \quad & (p+h)\xi_i - (c+h)x \geq t_i, \quad \forall i \in [N] \\ & (p-c)x \geq t_i, \quad \forall i \in [N] \\ & x \in \mathbb{Z}_+ \end{aligned}$$

### 4.2 Worst-case value

Next, we consider

$$\rho_f\left(x; \{\xi_i\}_{i\in[N]}\right) = \min_{i\in[N]} f(x,\xi_i).$$

Here, the worst-case value is defined as the **minimum** of $f(x, \xi_1), \ldots, f(x, \xi_N)$. Again, this is because we are trying to maximize the profit. Then we want to solve

$$\max_{x} \quad \min_{i \in [N]} f(x, \xi_i).$$

This can be reformulated as

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & f(x, \xi_i) \geq t, \quad \forall i \in [N], \\ & x \in \mathbb{Z}_+ \end{aligned}$$

Plugging in the formula $f(x, \xi) = \min\{(p+h)\xi - (c+h)x,\ (p-c)x\}$, we obtain

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & (p+h)\xi_i - (c+h)x \geq t, \quad \forall i \in [N] \\ & (p-c)x \geq t, \quad \forall i \in [N] \\ & x \in \mathbb{Z}_+ \end{aligned}$$

### 4.3 Conditional Value at Risk

We may reformulate the problem $\max_x f(x, \xi)$ as

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & f(x, \xi) \geq v. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & v - f(x, \xi) \leq 0. \end{aligned}$$

Then $v - f(x, \xi)$ can be viewed as a constraint function. Here, the Conditional Value at Risk at level $\alpha \in (0, 1)$ defined as

$$\begin{aligned} \mathrm{CVaR}_\alpha\left(v - f(x, \xi); \hat{P}_N\right) \quad = \quad \max \quad & \frac{1}{1-\alpha} \sum_{i=1}^{N} z_i \cdot (v - f(x, \xi_i)) \\ \text{s.t.} \quad & 0 \leq z_i \leq p_i \\ & \sum_{i=1}^{N} z_i = 1 - \alpha. \end{aligned}$$

By strong LP duality,

$$\begin{aligned} \mathrm{CVaR}_\alpha\left(v - f(x, \xi); \hat{P}_N\right) \quad = \quad \min \quad & t + \frac{1}{1-\alpha} \sum_{i=1}^{N} p_i r_i \\ \text{s.t.} \quad & r_i \geq 0, \quad i \in [N] \\ & t + r_i \geq v - f(x, \xi_i), \quad i \in [N]. \end{aligned}$$

Then

$$\mathrm{CVaR}_\alpha\left(v - f(x, \xi); \hat{P}_N\right) \leq 0$$

can be written as

$$
\begin{aligned}
t + \frac{1}{1-\alpha}\sum_{i=1}^{N} p_i r_i \le 0 \\
r_i \ge 0, \quad i \in [N] \\
t + r_i \ge v - f(x, \xi_i), \quad i \in [N].
\end{aligned}
$$

Then we solve

$$
\begin{aligned}
\max \quad & v \\
\text{s.t.} \quad & t + \frac{1}{1-\alpha}\sum_{i=1}^{N} p_i r_i \le 0 \\
& r_i \ge 0, \quad i \in [N] \\
& v - t - r_i \le f(x, \xi_i), \quad i \in [N] \\
& x \in \mathbb{Z}_+.
\end{aligned}
$$

Plugging in the formula $f(x, \xi) = \min\{(p+h)\xi - (c+h)x,\ (p-c)x\}$, we obtain

$$
\begin{aligned}
\max \quad & v \\
\text{s.t.} \quad & t + \frac{1}{1-\alpha}\sum_{i=1}^{N} p_i r_i \le 0 \\
& r_i \ge 0, \quad i \in [N] \\
& v - t - r_i \le (p+h)\xi - (c+h)x, \quad i \in [N] \\
& v - t - r_i \le (p-c)x, \quad i \in [N] \\
& x \in \mathbb{Z}_+.
\end{aligned}
$$