IE 331 OR: Optimization **KAIST, Spring 2023**
Lecture #17: Integer programming III May 2, 2023
Lecturer: Dabeen Lee

1 Outline

In this lecture, we cover

- disjunctive constraints,
- production with economic feasibillity,
- a variant of the facility location problem.

2 Disjunctive constraints

We wish to switch between two different constraints:

$$a^\top x \leq b, \quad c^\top x \leq d.$$

Since we switch between the two constraints, **at least one of them must be satisfied**. Therefore, we can write this as

$$a^\top x \leq b \quad \textbf{or} \quad c^\top x \leq d.$$

The "**or**" relationship is refer to as a **disjunction**.

How can we model this disjunction with linear constraints? We use a binary variable $y \in \{0,1\}$ to indicate which of the two constraints in the disjunction is satisfied. We model implications

$$\begin{aligned} y = 0 \quad &\Rightarrow \quad a^\top x \leq b \\ y = 1 \quad &\Rightarrow \quad c^\top x \leq d. \end{aligned}$$

Suppose that we know $a^\top x$ never exceeds $b + M_a$ for some $M_a > 0$. Suppose also that $c^\top x$ never exceeds $d + M_c$ for some $M_c > 0$. Then we may model the implications with the following linear constraints.

$$\begin{aligned} &a^\top x \leq b + M_a y \\ &c^\top x \leq d + M_c(1-y). \end{aligned}$$

This is another application of the big-M technique.

Note that "p or q" is logically equivalent to the statement

$$\neg p \quad \Rightarrow \quad q.$$

Therefore,

$$a^\top x \leq b + M_a y, \quad c^\top x \leq d + M_c(1-y)$$

is equivalent to

$$a^\top x > b \quad \Rightarrow \quad c^\top x \leq d.$$

The reason is that if $a^\top x > b$, then $a^\top x \leq b + M_a y$ forces $y = 1$. Then $c^\top x \leq d + M_c(1-y)$ and $y = 1$ induce $c^\top x \leq d$.

We may have systems of linear inequalities in a disjunction as follows.

$$Ax \le b \quad \textbf{or} \quad Cx \le d.$$

As before, we assume that there exist sufficiently large constants $M_A, M_C > 0$ such that

$$Ax \le b + M_A \mathbf{1}, \quad Cx \le d + M_C \mathbf{1}$$

hold where $\mathbf{1}$ is the vector of all ones. Then the following models the disjunction.

$$\begin{aligned} Ax &\le b + y \cdot M_A \mathbf{1}, \\ Cx &\le d + (1 - y) \cdot M_C \mathbf{1}, \\ y &\in \{0, 1\} \end{aligned}$$

3 Production with economic feasibility

Dorian Auto is considering manufacturing three types of automobiles: compact, midsize, and large cars. The resources required for and the profits yielded by each type of cars are shown in the following table.

Resource	Car type			Available resource
	Compact	Midsize	Large	
Steel	1.5 tons	3tons	5 tons	6,000 tons
Labor	30 hours	25 hours	40 hours	60,000 hours
Profit ($)	2,000	3,000	4,000	

Due to setup costs, if Dorian chooses to produce a certain type of car, it is only economically feasible to produce at least 1,000 cars of that type. Then we formulate an integer programming model to maximize Dorian's profit.

Decisions: Let x_1, x_2, x_3 denote the number of compact cars, the number of midsize cars, and the number of large cars that we produce. Hence thy are nonnegative integer variables.

Resource constraints: The current stock of steel is 6,000 tons, and that of labor resources is 60,000 hours. Therefore, we have

$$\begin{aligned} 1.5x_1 + 3x_2 + 5x_3 &\le 6000 \\ 30x_1 + 25x_2 + 40x_3 &\le 60000 \\ x_1, x_2, x_3 &\in \mathbb{Z}_+ \end{aligned}$$

Objective: We want to maximize the profit:

$$\max \quad 2000x_1 + 3000x_2 + 4000x_3.$$

Economic feasibility constraints: If we produce a nonzero quantity of car type i, then we need to produce at least 1000 units. Hence, we have the implication

$$x_i > 0 \quad \Rightarrow \quad x_i \ge 1000.$$

This is equivalent to

$$x_i \leq 0 \quad \textbf{or} \quad x_i \geq 1000.$$

As this is a disjunctive constraint, we may model this with linear constraints and binary variables. For each type i, we use binary variable $y_i \in \{0, 1\}$ and add constraints

$$\begin{aligned} x_i &\leq M_{i,1} y_i \\ -x_i &\leq -1000 + M_{i,2}(1 - y_i) \\ y_i &\in \{0, 1\}. \end{aligned}$$

What should $M_{i,1}$ be? From the first resource constraint, we have

$$1.5x_1 + 3x_2 + 5x_3 \leq 6000 \quad \Rightarrow \quad x_1 \leq 4000,\ x_2 \leq 2000,\ x_3 \leq 1200.$$

Therefore, we may set

$$M_{1,1} = 4000, \quad M_{2,1} = 2000, \quad M_{3,1} = 1200.$$

What should $M_{i,2}$ be? We know that $-x_i \leq 0$ for any i. Therefore, we may set

$$M_{i,2} = 1000, \quad i \in \{1, 2, 3\}.$$

Then $-x_i \leq -1000 + M_{i,2}(1 - y_i)$ becomes

$$x_i \geq 1000 y_i.$$

In summary, we deduce the following model.

$$\begin{aligned} \max \quad & 2000x_1 + 3000x_2 + 4000x_3 \\ \text{s.t.} \quad & 1.5x_1 + 3x_2 + 5x_3 \leq 6000 \\ & 30x_1 + 25x_2 + 40x_3 \leq 60000 \\ & 1000y_1 \leq x_1 \leq 4000y_1 \\ & 1000y_2 \leq x_2 \leq 2000y_2 \\ & 1000y_3 \leq x_3 \leq 1200y_3 \\ & x_1, x_2, x_3 \in \mathbb{Z}_+, \\ & y_1, y_2, y_3 \in \{0, 1\} \end{aligned}$$

4 Facility location revisited

Suppose that we have d different suburbs. We want to select some of these to be locations for fire stations.

- The expected number of yearly fire calls for each suburb j is given by e_j.
- The travel cost between location i and location j is c_{ij}.
- The recurring yearly cost of maintaining a fire station in a suburb j is f_j.

How can we decide which suburbs to place fire stations in to minimize the yearly cost?

Categorical decisions: Let x_j be a binary variable for each location $j \in [d]$.

$$x_j = \begin{cases} 1, & \text{if a fire station is located in suburb } j, \\ 0, & \text{no fire station in suburb } j. \end{cases}$$

Constraints: We do not have constraints on the number of fire stations for this particular problem.

Objective: Again, we want to minimize the yearly cost. Suppose that we have a fire station at location i and that the fire station serves suburb j. If a fire breaks out at location j, then it incurs travel cost of c_{ij} from fire station at i. Yearly, the expected cost is $c_{ij}e_j$ as e_j is the number of fires at location j. Moreover, suburb j pays f_j for the maintenance of its assigned fire station. Hence, the objective is

$$\min \quad \sum_{j \in [d]} \left(f_j x_j + e_j \sum_{i \in [d]} c_{ij} \cdot \mathbf{1}\left[\text{fire station at } i \text{ serves } j\right] \right)$$

For each suburb j, we select a fire station to serve it, for which we introduce binary variables

$$y_{ij} = \begin{cases} 1, & \text{if fire station at suburb } i \text{ serves suburb } j \\ 0, & \text{otherwise.} \end{cases}$$

One fire station is selected for each suburb, so we add

$$\sum_{i \in [d]} y_{ij} = 1.$$

As before, we add constraints

$$y_{ij} \leq x_i, \quad i, j \in [d]$$

to model implications $y_{ij} = 1 \Rightarrow x_i = 1$.

Then we can rewrite the travel cost for suburb j as

$$e_j \sum_{i \in [d]} c_{ij} y_{ij}.$$

Consequently, the complete model is given by

$$\begin{aligned} \min \quad & \sum_{j \in [d]} f_j x_j + \sum_{j \in [d]} \sum_{i \in [d]} e_j c_{ij} y_{ij} \\ \text{s.t.} \quad & \sum_{i \in [d]} y_{ij} = 1, \quad j \in [d] \\ & y_{ij} \leq x_i, \quad i, j \in [d] \\ & x \in \{0,1\}^d, \ y \in \{0,1\}^{d \times d} \end{aligned}$$

In fact, we do not need to impose the binary constraints $y \in \{0,1\}^{d\times d}$. Instead, we add constraints $y_{ij} \geq 0$ for $i, j \in [d]$. Then

$$\begin{aligned} \min \quad & \sum_{j\in[d]} f_j x_j + \sum_{j\in[d]} \sum_{i\in[d]} e_j c_{ij} y_{ij} \\ \text{s.t.} \quad & \sum_{i\in[d]} y_{ij} = 1, \quad j \in [d] \\ & 0 \leq y_{ij} \leq x_i, \quad i, j \in [d] \\ & x \in \{0,1\}^d. \end{aligned}$$

Solving this model would automatically impose $y \in \{0,1\}^{d\times d}$.