1 Outline

In this lecture, we cover

- the minimum cost flow model in matrix form,
- totally unimodular matrices.

2 Minimum cost flow model in matrix form

Recall that the minimum cost flow problem over network $D = (N, A)$ is modeled as the following linear program.

$$\begin{aligned} \min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\ \text{s.t.} \quad & \sum_{j\in N:(i,j)\in A} x_{ij} - \sum_{k\in N:(k,i)\in A} x_{ki} = b_i, \quad i \in N \\ & \ell \le x \le u. \end{aligned}$$

Given a network $D = (N, A)$, the **node-arc incidence matrix** M has $|N|$ rows corresponding to the nodes and $|A|$ columns corresponding to the arcs. The entries of M is given by

$$m_{i,(k,j)} = \begin{cases} 1, & \text{if } k = i, \\ -1, & \text{if } j = i, \\ 0, & \text{if } k \neq i \text{ and } j \neq i \end{cases}$$

for any $i \in N$ and $(k, j) \in A$. For example, consider the network over 6 nodes in Figure 12.1. We

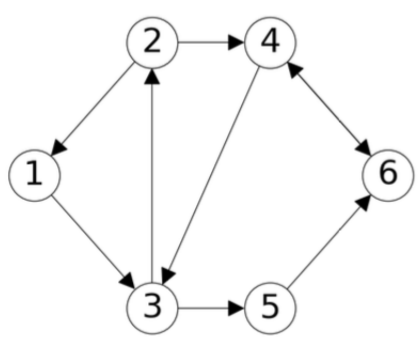

Figure 12.1: Network over 6 nodes

have

$$N = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad A = \{(1, 3), (2, 1), (2, 4), (3, 2), (3, 5), (4, 3), (4, 6), (5, 6), (6, 4)\}.$$

In this case, the incidence matrix is given as the following table.

$m_{i,(k,j)}$	$(1,3)$	$(2,1)$	$(2,4)$	$(3,2)$	$(3,5)$	$(4,3)$	$(4,6)$	$(5,6)$	$(6,4)$
1	1	-1							
2		1	1	-1					
3	-1			1	1	-1			
4			-1			1	1		-1
5					-1			1	
6							-1	-1	1

Let b be the vector with entries b_i for $i \in N$. Then we may write the flow balance constraints as

$$Mx = b.$$

The ith row of this matrix equation is

$$\sum_{(k,j)\in A} m_{i,(k,j)} x_{kj} = b_i.$$

Here, the left-hand side is given by

$$\begin{aligned}
\sum_{(k,j)\in A} m_{i,(k,j)} x_{kj} &= \sum_{(k,j)\in A:k=i} m_{i,(k,j)} x_{kj} + \sum_{(k,j)\in A:j=i} m_{i,(k,j)} x_{kj} \\
&= \sum_{(k,j)\in A:k=i} x_{kj} - \sum_{(k,j)\in A:j=i} x_{kj} \\
&= \sum_{j\in N:(i,j)\in A} x_{ij} - \sum_{k\in N:(k,i)\in A} x_{ki}.
\end{aligned}$$

Therefore, $Mx = b$ indeed collects the set of flow balance constraints.

The incidence matrix has the following properties.

- Entries are -1, 0, and $+1$ only.
- Each column has only two nonzero entries, $+1$ and -1.
- The column for arc (k,j) has $+1$ in row k and -1 in row j.
- Adding up all rows of M, we obtain a row of all zeros, i.e., $\mathbf{1}^\top Mx = 0$.

Remark 12.1. If $Mx = b$ is feasible, then

$$0 = \mathbf{1}^\top Mx = \mathbf{1}^\top b = \sum_{i\in N} b_i.$$

Therefore, if $\sum_{i\in N} b_i \neq 0$, then $Mx = b$ is infeasible.

With the incidence matrix, the minimum cost flow model can be written as

$$\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\text{s.t.} \quad & Mx = b \\
& \ell \leq x \leq u.
\end{aligned}$$

Theorem 12.2. *Let M be the node-arc incidence matrix of a network. Consider any linear program with constraints*
$$Mx = b, \quad \ell \le x \le u.$$
Suppose that b, ℓ, u have only integer entries. Then there exists an optimal solution x^* to the linear program that has only integer entries.

Next let us consider a more general model. The flow balance constraint imposes that the net supply is equal to a prescribed value b_i, i.e.,

$$\text{net-supply}(i;x) = \text{outflow}(i;x) - \text{inflow}(i;x) = \sum_{j\in N:(i,j)\in A} x_{ij} - \sum_{k\in N:(k,i)\in A} x_{ki} = b_i.$$

In general, we allow an interval of values $\left[b_i^L, b_i^G\right]$, and as long as the net supply is in the interval, x is feasible. This can be written as

$$b_i^L \le \text{net-supply}(i;x) = \sum_{j\in N:(i,j)\in A} x_{ij} - \sum_{k\in N:(k,i)\in A} x_{ki} \le b_i^G.$$

- $b_i^L = -\infty$ is equivalent to having only the upper bound net-supply$(i;x) \le b_i^G$.
- $b_i^G = +\infty$ is equivalent to having only the lower bound net-supply$(i;x) \ge b_i^L$.
- $b_i^L = -\infty$ & $b_i^G = +\infty$ is equivalent to imposing no constraint on net-supply$(i;x)$.

The general model can be written as

$$\begin{aligned} \min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\ \text{s.t.} \quad & b_i^L \le \sum_{j\in N:(i,j)\in A} x_{ij} - \sum_{k\in N:(k,i)\in A} x_{ki} \le b_i^G, \quad i \in N \\ & \ell \le x \le u. \end{aligned}$$

Let b^L be the vector of entries b_i^L for $i \in N$, and let b^G be the vector of entries b_i^G for $i \in N$. With the incidence matrix M,

$$\begin{aligned} \min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\ \text{s.t.} \quad & b^L \le Mx \le b^G, \\ & \ell \le x \le u. \end{aligned}$$

When $b^L = b^G = b$, this reduces to the first minimum cost flow model.

Theorem 12.3. *Let M be the node-arc incidence matrix of a network. Consider any linear program with constraints*
$$b^L \le Mx \le b^G, \quad \ell \le x \le u.$$
Suppose that b^L, b^G, ℓ, u have only integer entries. Then there exists an optimal solution x^* to the linear program that has only integer entries.

3 Totally unimodular matrices

Let M be an $m \times d$ matrix. A **submatrix** of M is a matrix that consists of the entries in a subset of rows and a subset of columns. For example, we take nodes $\{2, 4, 6\}$ and columns $\{(2, 4), (4, 6), (6, 4)\}$.

$m_{i,(k,j)}$	$(1,3)$	$(2,1)$	$(2,4)$	$(3,2)$	$(3,5)$	$(4,3)$	$(4,6)$	$(5,6)$	$(6,4)$
1	1	-1							
2		1	1	-1					
3	-1			1	1	-1			
4			-1			1	1		-1
5					-1			1	
6							-1	-1	1

Then the corresponding submatrix of the node-arc incidence matrix is given by

	$(2,4)$	$(4,6)$	$(6,4)$
2	1	0	0
4	-1	1	-1
6	0	-1	1

A **square submatrix** of M is a submatrix of M that is a square matrix, i.e., the number of rows and that of columns are equal.

A matrix M is **totally unimodular** if every square submatrix of M has determinant $-1, 0, 1$ only. Note that each entry itself is an 1×1 square submatrix, so if M is totally unimodular, all its entries are $-1, 0, 1$ only.

Theorem 12.4. *Let M be the node-incidence matrix of a network, and let I be the identity matrix that has the same number of columns as M. Then*

$$\begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix}$$

is totally unimodular.

Note that by definition, if a matrix is totally unimodular, then all its submatrices are totally unimodular. In particular, the node-arc incidence matrix M itself is totally unimodular.

Theorem 12.5 (Hoffman and Kruskal [HK56])**.** *Consider a linear program given by*

$$\begin{aligned} \min \quad & c^\top x \\ s.t. \quad & Px \leq b. \end{aligned}$$

If P is totally unimodular and b has integer entries only, then there exists an optimal solution x^* to the linear program that has only integer entries.

We now prove Theorems 12.2 and 12.3. In fact, it suffices to prove Theorem 12.3, as the generalized minimum cost flow model implies the original model.

Proof of Theorem 12.3. The generalized minimum cost flow model is given by

$$\begin{aligned} \min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\ \text{s.t.} \quad & b^L \leq Mx \leq b^G, \\ & \ell \leq x \leq u. \end{aligned}$$

We may write the constraints as

$$\begin{aligned} Mx &\leq b^G \\ -Mx &\leq -b^L \\ x &\leq u \\ -x &\leq -\ell \end{aligned}$$

Then the constraints can be taken into the following matrix inequality form.

$$\begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix} x \leq \begin{bmatrix} b^G \\ -b^L \\ u \\ -\ell \end{bmatrix}.$$

By Theorem 12.4, the resulting constraint matrix is totally unimodular. As $b^G, -b^L, u, -\ell$ have all integer entries, it follows from Theorem 12.5 that there is an optimal solution x^* that has integer entries only. □

References

[HK56] A.J. Hoffman and J.B. Kruskal. Integral boundary points of convex polyhedra. In Kuhn H.W. and Tucker A.W., editors, *Linear inequalities and related systems, Ann. Math. Studies*, volume 38, pages 223–246. 1956. 12.5