Lecture 8: the matching polytope and separation

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Outline

- Matching polytope
- Ellipsoid algorithm and its consequences in combinatorial optimization
- Separation-based approach for matching

LP formulation for matching

- Recall that a maximum weight matching in a graph $G = (V, E)$ with weights $w \in \mathbb{R}^{|E|}$ can be computed by solving

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} w_e x_e \\ \text{subject to} \quad & \sum_{v \in V : uv \in E} x_{uv} \leq 1 \quad \text{for all } u \in V, \\ & x_e \in \{0, 1\} \quad \text{for all } e \in E. \end{aligned} \tag{1}$$

LP formulation for matching

- Moreover, when G is bipartite, our approach was to take its LP relaxation

$$
\begin{aligned}
\text{maximize} \quad & \sum_{e \in E} w_e x_e \\
\text{subject to} \quad & \sum_{v \in V : uv \in E} x_{uv} \leq 1 \quad \text{for all } u \in V, \\
& x_e \geq 0 \quad \text{for all } e \in E.
\end{aligned}
\tag{2}
$$

Fractionality from an odd cycle

- Unlike the bipartite case, solving (2) when G is not bipartite does not give us a maximum weight matching.

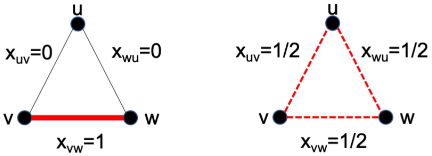


Figure: factionality of the linear programming relaxation

Matching polytope

- The matching polytope of a graph G is formally defined as the convex hull of the incidence vectors of matchings in G
- The convex hull is the set of solutions satisfying the constraints of (1).
- Hence, the matching polytope is given by

$$P_{\text{matching}}(G) = \mathrm{conv}\left\{ x \in \{0,1\}^{|E|} : \sum_{v \in V : uv \in E} x_{uv} \leq 1 \quad \text{for all } u \in V \right\}.$$

- We argued that the formulation (1) for the maximum weight bipartite matching problem is equivalent to

$$\max \left\{ \sum_{e \in E} w_e x_e : \ x \in P_{\text{matching}}(G) \right\}. \tag{3}$$

Matching polytope

Proposition

Let $G = (V, E)$ be a bipartite graph. Then

$$P_{\text{matching}}(G) = \left\{ x \in [0,1]^{|E|} : \sum_{v \in V : uv \in E} x_{uv} \leq 1 \quad \text{for all } u \in V \right\}.$$

Matching polytope

- For a nonbipartite graph, the example implies that the degree constraints are not enough to characterize the matching polytope.
- We next explain additional inequalities that are necessary to describe the matching polytope.
- Let $U \subseteq V$ be a subset of the vertex set with an odd number of vertices.

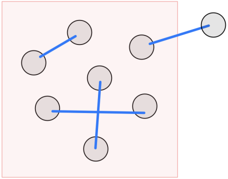
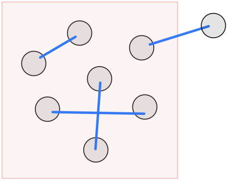


Figure: odd cardinality subset

Matching polytope



- Then look at the set of edges that are fully contained in U.
- Then the following inequality is satisfied by any solution to the integer program:

$$\sum_{e \in E(U)} x_e \leq \frac{|U| - 1}{2}$$

  where $E(U)$ is the set of edges fully contained in U.
- We call this inequality an **odd-set inequality**.

Matching polytope

- Validity of

$$\sum_{e \in E(U)} x_e \leq \frac{|U| - 1}{2}$$

where $E(U)$ is the set of edges fully contained in U.

Matching polytope

- For the triangle, note that the $U = \{u, v, w\}$ is an odd cardinality subset, and the corresponding odd-set inequality is $x_{uv} + x_{vw} + x_{wu} \leq 1$.
- Hence, imposing the odd-set inequality, we may exclude the fractional solution $(x_{uv}, x_{vw}, x_{wu}) = (1/2, 1/2, 1/2)$.

Theorem (Edmonds)

Let $G = (V, E)$ be a graph, not necessarily bipartite. Then

$$P_{matching}(G) = \left\{ x \in [0,1]^{|E|} : \begin{array}{ll} \displaystyle\sum_{v \in V : uv \in E} x_{uv} \leq 1 & \text{for all } u \in V, \\ \displaystyle\sum_{e \in E(U)} x_e \leq \frac{|U| - 1}{2} & \text{for all } U \subseteq V \text{ with } |U| \geq 3 \text{ odd} \end{array} \right\}.$$

Ellipsoid algorithm

- We introduce the ellipsoid algorithm.
- The problem that we consider is as follows.

  Given a polyhedron $P = \{x \in \mathbb{R}^d : Ax \leq b\}$,
  (1) conclude that the interior of P *is empty, or*
  (2) find a point $\bar{x}$ *contained in the interior of* P.

- This is a variant of the **feasibility problem**.

Ellipsoid algorithm

Algorithm 1 Ellipsoid algorithm

Initialize a polyhedron $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ and a sufficiently large ellipsoid E_1.

for $t = 1, \ldots, T$ **do**

  if the center x^t of ellipsoid E_t satisfies $Ax^t < b$ **then**

    Stop and conclude that the interior of P contains x^t.

  else

    There exists some inequality $\alpha^\top x \leq \beta$ in the system $Ax \leq b$ such that $\alpha^\top x^t \geq \beta$.

    Let E_{t+1} be the smallest ellipsoid containing $E_t \cap \left\{x \in \mathbb{R}^d : \alpha^\top x \leq \beta\right\}$.

    $t \rightarrow t + 1$.

  end if

  Conclude that the interior of P is empty.

end for

Ellipsoid algorithm

Theorem (Kachyan)

The ellipsoid algorithm (Algorithm 1) terminates with a correct answer if E_1 and T are properly chosen.

- In fact, Kachyan showed that one can choose E_1 and T so that their encoding sizes are polynomially bounded, in which case Algorithm 1 runs in polynomial time.

Ellipsoid algorithm

- The important part is that the ellipsoid algorithm can be turned into a polynomial algorithm for the problem of optimizing a linear function over P.
- The idea is based on binary search.
- Basically, if we want to minimize a linear function $c^\top x$, then we consider

$$\left\{x \in \mathbb{R}^d : \ Ax \leq b, \ c^\top x \leq v\right\}$$

for varying v.

Ellipsoid algorithm

Theorem (Kachyan)

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Equivalence between optimization and separation

- Next we formally state the **equivalence between optimization and separation**.

- Let $P \subseteq \mathbb{R}^d$ be a rational polytope such that

$$P = \operatorname{conv}\{v^1, \ldots, v^n\}.$$

- Then we say that $P \subseteq \mathbb{R}^d$ belongs to a **well-described family of rational polyhedra** if the length L of input needed to describe P satisfies $d \le L$ and $\log D$ is bounded by a polynomial function of L, where D is the largest numerator or denominator of the rational vectors v^k for $k \in [n]$ and $h \in [\ell]$.

- Here, we care about the number D to bound the complexity of the ellipsoid method.

Equivalence between optimization and separation

1. **Separation Problem**
   Given a well-defined polyhedron $P \subseteq \mathbb{R}^d$ and $\bar{x} \in \mathbb{Q}^d$, either show that $\bar{x} \in P$ or find an inequality $\alpha^\top x \leq \beta$ satisfied by all $x \in P$ such that $\alpha^\top \bar{x} > \beta$.

2. **Optimization Problem**
   Given a well-defined polyhedron $P \subseteq \mathbb{R}^d$ and $c \in \mathbb{Q}^d$, find x^* such that $c^\top x^* = \max\{c^\top x : \ x \in P\}$ or show that $P = \emptyset$.

Theorem (Grötschel, Lovász, and Schrijver)

For a well-defined polyhedron P, the separation can be solved in polynomial time if and only if the optimization problem can be solved in polynomial time.

Matching from separation

- We solve

$$\max\left\{\sum_{e\in E} w_e x_e \;:\; x \in P_{\text{matching}}(G)\right\},$$

which is given by

$$\begin{aligned}
\text{maximize} \quad & \sum_{e\in E} w_e x_e \\
\text{subject to} \quad & \sum_{v\in V: uv\in E} x_{uv} \leq 1 \quad \text{for all } u \in V, \\
& \sum_{e\in E(U)} x_e \leq \frac{|U|-1}{2} \quad \text{for all } U \subseteq V \text{ with } |U| \geq 3 \text{ odd}, \\
& x_e \geq 0 \quad \text{for all } e \in E.
\end{aligned} \tag{4}$$

Matching from separation

- Although (4) is a linear program, one issue is that the number of odd cardinality subsets of V can be exponential in $|V|$.
- In that case, writing down all odd-set inequalities for (4) cannot be done in polynomial time.
- Nevertheless, the optimization problem (4) is shown to be solvable in polynomial time by the equivalence between separation and optimization.

Matching from separation

- To show that (4) can be solved in polynomial time, we show that the separation problem over the matching polytope $P_{\text{matching}}(G)$ can be solved in polynomial time.
- Given $\bar{x} \in \mathbb{Q}^{|E|}$, we want to decide whether $\bar{x} \in P_{\text{matching}}(G)$ or find an inequality $\alpha^\top x \leq \beta$ that separates $\bar{x}$ from $P_{\text{matching}}(G)$.
- For the matching polytope, we can check whether $\bar{x}$ satisfies the degree constraints and the nonnegativity constraints in $O(|V| + |E|)$ time.
- Hence, the question is as to whether we can decide that $\bar{x}$ satisfies the odd-set inequalities in polynomial time.
- In fact, the separation problem can be solved in polynomial time with its connection to the so-called **minimum odd cut problem**.