Lecture 6: extensions of bipartite matching

Dabeen Lee

Industrial and Systems Engineering, KAIST

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Outline

- Stable matching
- Online bipartite matching

Doctor-Hospital Assignment

- Let us recall the doctor-hospital assignment scenario for the US medical system.

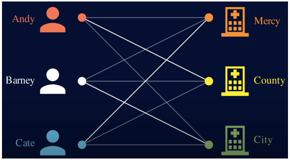


Figure: doctor-hospital assigment

- One may associate it with a bipartite network between a list of medical doctors and a list of hospitals.
- We assume that a hospital has at most one position available.
- Then we can imagine that the assignment problem can be solved by bipartite matching.

Doctor-Hospital Assignment

Preferences

- In real world scenarios, however, doctors have their preferences over certain hospitals.
- At the same time, it is common for hospitals to set priorities over candidates with certain specialties.

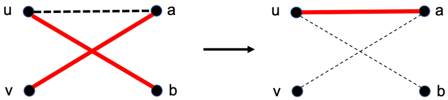
Stable matching

- Take a bipartite graph $G = (V, E)$ where the vertex set V is decomposed into D and H where D represents doctors and H is for hospitals.
- Individual doctors in D have a ranking of the hospitals of H based on their preferences.
- Similarly, individual hospitals in H have a ranking of the doctors in D based on their priorities.
- Essentially, we want to compute **a matching** between doctors and hospitals, **taking into account the rankings**.

Stable matching

Unstable pair

- The goal of this section is to find a matching without an **unstable pair**, which is called a **stable matching**.
- Suppose that a doctor u is matched to a hospital b and a doctor v is matched to a hospital a.



- Imagine a situation when doctor u prefers hospital a over hospital b and at the same time, hospital a also prefers doctor u over doctor v.
- Then doctor u and hospital a have an incentive to break their current assignments and start a new contract between them.
- **In this case, we call (u, a) an unstable pair.**

Gale-Shapley algorithm

- In 1962, David Gale and Lloyd Shapley propsed an algorithm for finding a stable matching.
- The algorithm is now known as the **Gale-Shapley algorithm** or the **propose-and-reject algorithm**.

Algorithm

1. Each doctor applies to the hospital that is on the top of the preference ranking which has not previoulsy rejected the doctor.
2. Each hospital rejects all applicants except for the top candidate and keeps the candidate until a better one applies.
3. Repeat steps 1–3 until every doctor either has been linked to a hospital or has been rejected from all hospitals on the preference list.

Correctness

Theorem

The Gale-Shapley algorithm correctly finds a stable matching in $O(|V|^2)$ *iterations.*

LP formulation for stable matching

- Recall the maximum weight bipartite matching formulation without the stability condition:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{e \in E} w_e x_e \\
\text{subject to} \quad & \sum_{v \in V : uv \in E} x_{uv} \leq 1 \quad \text{for all } u \in V, \\
& x_e \geq 0 \quad \text{for all } e \in E.
\end{aligned} \tag{1}
$$

- To avoid unstability between doctor u and hospital a, we need to write a constraint.

LP formulation for stable matching

Given two edges $e, f \in E$, we say that f **precedes** e if they satisfy the following conditions.

- e and f share a common end point.
- If $e = uy$ and $f = ux$, then u prefers x over y.
- If $e = vx$ and $f = ux$, then x preferx u over v.

LP formulation for stable matching

- Hence, f precedes e if the connection f has a higher priority over the connection e.
- When f precedes e, we express it as $f \succeq e$.
- Then $e \succeq e$ trivially holds.
- Vande Vate in 1989 observed that for any $e \in E$, unstability for e can be avoided by imposing

$$\sum_{f \in E : f \succeq e} x_f \geq 1. \tag{2}$$

LP formulation for stable matching

Validity of the inequality

- Suppose that $e = ux \in E$ ends being unstable.
- Then there exist $uy, vx \in E$ such that $uy \not\succeq ux$ and $vx \not\succeq ux$ while $x_{uy} = x_{vx} = 1$.
- This means that

$$\sum_{z \in H: uz \succeq ux} x_{uz} \leq 1 - x_{uy} = 0,$$

$$\sum_{w \in D: wx \succeq ux} x_{wx} \leq 1 - x_{vx} = 0.$$

- This in turn implies that

$$\sum_{f \in E: f \succeq e} x_f = 0 \not\geq 1,$$

  violating the constraint (2).
- Therefore, imposing (2) would let us avoide any unstable pair.

LP formulation for stable matching

Completeness

- Vande Vate in 1989 further proved that the linear program with (2) given by

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} w_e x_e \\ \text{subject to} \quad & \sum_{v \in V : uv \in E} x_{uv} \leq 1 \quad \text{for all } u \in V, \\ & \sum_{f \in E : f \succeq e} x_f \geq 1 \quad \text{for all } e \in E, \\ & x_e \geq 0 \quad \text{for all } e \in E \end{aligned} \tag{3}$$

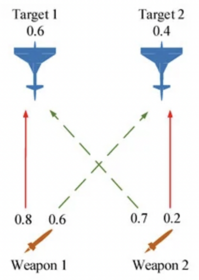
returns a maximum weight stable matching.

Online bipartite matching

- So far, one of the inherent assumptions was that the entire structure of a given bipartite graph is available to the decision-maker.
- Hence, an algorithm receives the entire graph and computes a matching that is globally optimal.
- In many real world applictions, only some local structures of the graph is accessible while others are revealed gradually over time.

Online bipartite matching

Weapon-target assignment



- One side prepares a missile defense system while the other side launches fighter aircrafts.
- It is quite rare that all enemy jets arrive at the same time, while it is more common that they arrive in an unpredictable sequence.
- To defend against an enemy fighter, we would have to assign a missile to it in real time.

Online bipartite matching

- We consider the so-called **online bipartite matching** problem.
- Take a bipartite graph $G = (V, E)$ where the vertex set V is partitioned into V_1 and V_2.
- At the beginning, the vertex set V_1 is present.
- In contrast, the vertices in V_2 arrive **online**, which means that the vertices arrive one by one in a sequence while the sequence is not known.
- When a vertex v in V_2 arrives, we may take its neighbor u in V_1 to match with it or we may decide to just skip it.

Competitive ratio

- An algorithm for online bipartite matching is evaluated by the size of the matching obtained after all vertices of V_2 arrive.
- An algorithm makes decisions only with local information about the graph, so the size of the final matching cannot be better than the maximum size of a matching in G.
- Nevertheless, our performance measure is the **competitive ratio** defined as

$$\frac{\text{The size of a matching constructed by algorithm } \mathcal{A}}{\text{The maximum size of a matching in } G}.$$

Greedy algorithm for online bipartite matching

Greedy algorithm

- Every time a vertex v in V_2 arrives, match it to one of its available neighbors.

Proposition

The simple greedy algorithm achieves a competitive ratio of $1/2$ for online bipartite matching.

Ranking algorithm for online bipartite matching

Ranking algorithm by Karp, Vazirani, and Vazirani (1990)

1. For each vertex $u \in V_1$, sample a weight $p_u \in [0, 1]$ uniformly at random.
2. Whenever a vertex $v \in V_2$ arrives, match v to its available neighbor that has the highest weight.

Ranking algorithm for online bipartite matching

- The simple algorithm achieves a better performance in expectation.
- To be more precise, we consider the notion of **expected competitive ration** defined as

$$\frac{\text{The \textbf{expected} size of a matching constructed by algorithm } \mathcal{A}}{\text{The maximum size of a matching in } G}.$$

Theorem (Karp, Vazirani, and Vazirani)

The ranking algorithm achieves an expected competitive ratio of $(1 - 1/e)$ *for online bipartite matching.*

- Here, $1 - 1/e$ is roughly 0.6321.
- Although the ranking algorithm is simple, its analysis is not as trivial.