Lecture 5: the Hungarian algorithm and matching markets

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Outline

- Hungarian algorithm for maximum weight bipartite matching
- Vickrey–Clarke–Groves pricing mechanism for matching markets

Combinatorial algorithm for maximum weight bipartite matching

- In Lecture 3, we learned an LP-based algorithm for maximum weight bipartite matching.
- Net we cover a combinatorial algorithm, that is known as the **Hungarian algorithm**.

Preprocessing step

1. First, as we are interested in a maximum weight matching, we may discard edges with a negative weight.
2. Up to adding dummy vertices and dummy edges with weight zero, we obtain a complete bipartite graph $K_{n,n}$ for some $n \geq 1$.

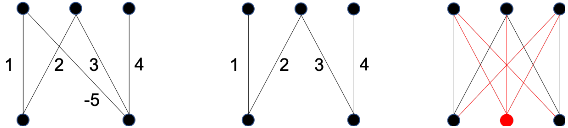


Figure: illustrating the preprocessing step

Hungarian algorithm

- After the preprocessing step, we may assume that $G = K_{n,n}$ for some $n \geq 1$ and $w \in \mathbb{R}_+^{|E|}$.
- Then the problem boils down to finding a **maximum weight perfect matching** in G.
- As before, let the vertex set V be partitioned into V_1 amd V_2 with $|V_1| = |V_2| = n$.
- Then a maximum weight matching in G can be computed by

$$
\begin{aligned}
\text{maximize} \quad & \sum_{e \in E} w_e x_e \\
\text{subject to} \quad & \sum_{v \in V_2} x_{uv} \leq 1 \quad \text{for all } u \in V_1, \\
& \sum_{u \in V_1} x_{uv} \leq 1 \quad \text{for all } v \in V_2, \\
& x_e \geq 0 \quad \text{for all } e \in E.
\end{aligned}
\tag{1}
$$

Hungarian algorithm

- Again, as $w_e \geq 0$ for all $e \in E$ and G is a complete bipartite graph, (1) has an optimal solution that corresponds to a perfect matching.
- Then it follows that (1) is equivalent to

$$\begin{aligned}
\text{maximize} \quad & \sum_{e \in E} w_e x_e \\
\text{subject to} \quad & \sum_{v \in V_2} x_{uv} = 1 \quad \text{for all } u \in V_1, \\
& \sum_{u \in V_1} x_{uv} = 1 \quad \text{for all } v \in V_2, \\
& x_e \geq 0 \quad \text{for all } e \in E.
\end{aligned} \tag{Primal}$$

- The dual of (Primal) is given by

$$\begin{aligned} \text{minimize} \quad & \sum_{u \in V_1} y_u + \sum_{v \in V_2} z_v \\ \text{subject to} \quad & y_u + z_v \geq w_{uv} \quad \text{for all } uv \in E. \end{aligned} \tag{Dual}$$

- The following result is a direct consequence of the **complementary slackness condition** for linear programming.

Lemma

Let M be a perfect matching in G, feasible to (Primal). Suppose that there exists a feasible solution (y, z) to (Dual) that satisfies $y_u + z_v = w_{uv}$ for every $uv \in M$. Then M is a maximum weight matching.

Hungarian algorithm

- Based on the lemma, the main idea behind the Hungarian algorithm is as follows.
  - (y, z) always remains feasible to (Dual), satisfying the constraints of (Dual).
  - Only an edge $uv \in E$ satisfying $y_u + z_v = w_{uv}$ can be added to our matching M.
- Once M becomes a perfect matching, becoming feasible to (Primal), then it will satisfy the conditions of the lemma, which guarantees that M is a maximum weight matching.

Hungarian algorithm

- To implement this idea, we introduce the notion of **equality subgraphs**.
- Given a feasible solution (y, z) to (Dual), we define the subgraph of G taking the edges $uv \in E$ satisfying $y_u + z_v = w_{uv}$.
- We use notation $G_{y,z}$ to denote the equality subgraph of G associated with (y, z).
  - Given a feasible solution (y, z) to (Dual), we take a maximum matching M in $G_{y,z}$.

Hungarian algorithm

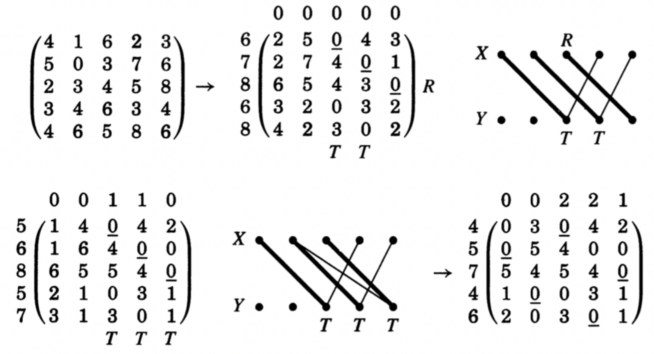
Hungarian algorithm

Algorithm 1 Hungarian algorithm for maximum weight bipartite matching

Input: complete bipartite graph $G = (V, E)$ with $V = V_1 \cup V_2$ and $w \in \mathbb{R}_+^{|E|}$
Initialize $y_u = \max_{v \in V_2} w_{uv}$ for $u \in V_1$, $z_v = 0$ for $v \in V_2$
Initialize $M = \emptyset$ and $B = \emptyset$
while M is not a perfect matching **do**
  Construct the equality subgraph $G_{y,z}$ associated with (y, z)
  Set M and B as a maximum matching and a minimum vertex cover in $G_{y,z}$, respectively
  Set $R = V_1 \cap B$ and $T = V_2 \cap B$
  Compute $\epsilon = \min\{y_u + z_v - w_{uv} : u \in V_1 - R, v \in V_2 - T\}$
  Update $y_u = y_u - \epsilon$ for $u \in V_1 - R$ and $z_v = z_v + \epsilon$ for $v \in T$
end while
Return M

Example

Let us consider an example with $G = K_{5,5}$.



In each matrix, the rows correspond to the vertices in V_1, and the columns are for the vertices in V_2.

Correctness

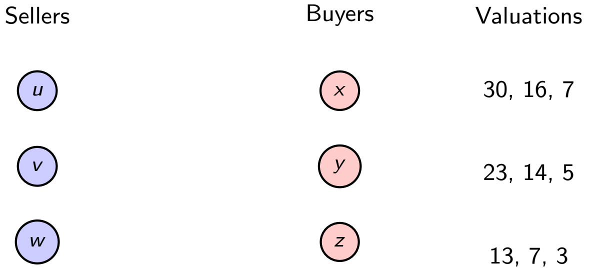
Theorem

Let $G = (V, E)$ be a complete bipartite graph, and let $w \in \mathbb{R}_+^{|E|}$. Then Algorithm 1 finds a maximum weight pefect matching in G.

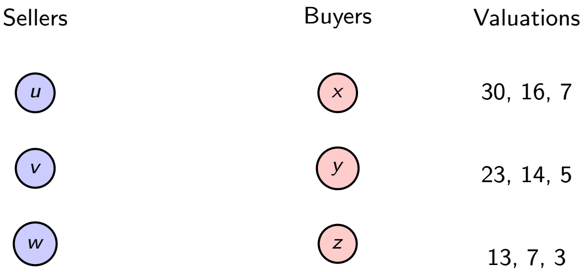
Correctness

Matching markets

- We have a nework of sellers and buyers for certain items in a market place.
- To simplify our discussion, let us assume that there are three sellers labeled u, v, and w and that we have a set of three buyers labeled x, y, and z.
- Each seller offers an item, and each buyer has certain valuations of the items.



Matching markets



- The sellers, or the market, are supposed to set the prices of items.
- For the item offered by seller $i \in \{u, v, w\}$, we use notation p_i for its price.
- We use notation v_{ij} to denote the valuation of buyer $j \in \{x, y, z\}$ for the item offered by seller $i \in \{u, v, w\}$.
- Then the **utility** of buyer j buying the item of seller i is given by

Matching markets

- We assume that the **rational behavior** of buyer j, which means that the buyer would decide to buy the item from seller i only if u_{ij} is nonnegative.
- It is natural that the assignment of buyers to sellers can be represented as a **bipartite matching**.
- Let $M \subseteq \{u, v, w\} \times \{x, y, z\}$ denote a matching or an assignment of buyers and sellers.
- Then the **social welfare** is defined as

the social welfare = the total profit of sellers + the total profit of buyers.

- Then it follows that

$$\textbf{the social welfare} = \sum_{ij \in M} (\textbf{the profit of buyer } i + \textbf{the profit of seller } j)$$
$$=$$
$$=$$

Matching markets

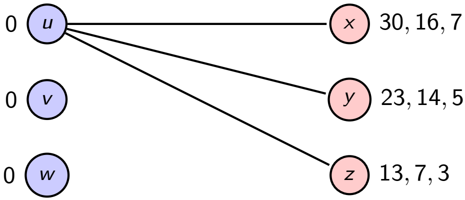
- Therefore, the social welfare equals the valuation sum of items that are matched with buyers.
- Then the social welfare can be viewed as the weight of a matching M where each assignment between seller i and buyer j is given by the item valuation v_{ij}.
- In turn, this implies that the social welfare is maximized if the corresponding matching is a maximum weight matching.

Matching markets

- However, individual buyers would behave rationally, so they will always target an item with the highest utility.
- It is quite likely to have conflicts between buyers.
- Then a market moderator would set a high price for a popular item.
- We call the set of prices are **market clearing** when a perfect matching is available under the prices.
- We will explain the **Vickrey–Clarke–Groves (VCG) mechanism** that is proven to be market clearing.

The VCG mechanism

- The basic idea is that whenever there is a conflict which forbids a perfect matching, we increase the price of some item.
- Here, a conflict can be captured by the notion of **preferred-seller graph**.
- For each buyer j, we draw an edge between buyer j and seller u for every $u \in \arg\max \{u_{ij} = v_{ij} - p_i : i \in \{u, v, w\}\}$.



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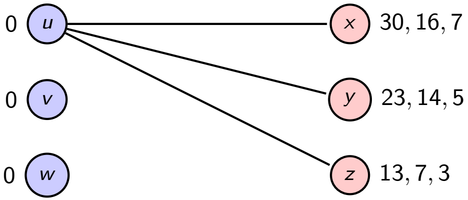
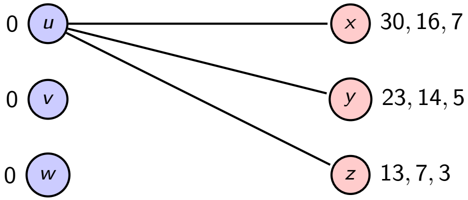


Matching markets



Matching markets

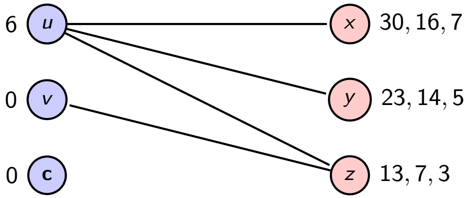


Figure: after increasing the price of the item in $N(S_1)$

Matching markets

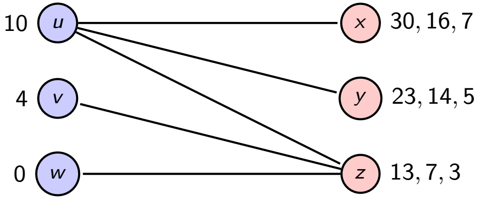


Figure: after increasing the prices of the items in $N(S_2)$

Matching markets

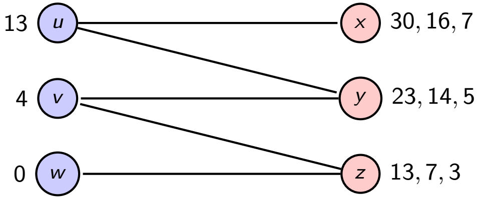


Figure: after increasingthe prices of the items in $N(S_3)$

Matching markets

Theorem

The Vickrey–Clarke–Groves (VCG) mechanism always finds a market clearing price that maximizes the social welfare in finite time.